https://www.petertanner.dev/posts/Idiots-guide-to-ELEC4402-Communications-Systems/

Fourier transform identities and properties

Time domain $x(t)$	Frequency domain $X(f)$
$\text{rect}\left(\frac{t}{T}\right) \quad \Pi\left(\frac{t}{T}\right)$	$T\text{sinc}(fT)$
$\text{sinc}(2Wt)$	$\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right) \quad \frac{1}{2W}\Pi\left(\frac{f}{2W}\right)$
$\exp(-at)u(t), \quad a>0$	$\frac{1}{a+j2\pi f}$
$\exp(-a\|t\|), \quad a>0$	$\frac{2a}{a^2+(2\pi f)^2}$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$
$1-\frac{\|t\|}{T}, \quad \|t\|<T \quad \text{tri}(t/T)$	$T\text{sinc}^2(fT)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$\exp(-j2\pi f t_0)$
$\exp(j2\pi f_c t)$	$\delta(f-f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]$
$\cos(2\pi f_c t+\theta)$	$\frac{1}{2}[\delta(f-f_c)\exp(j\theta)+\delta(f+f_c)\exp(-j\theta)] \quad \text{Use for coherent recv.}$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f-f_c)-\delta(f+f_c)]$
$\sin(2\pi f_c t+\theta)$	$\frac{1}{2j}[\delta(f-f_c)\exp(j\theta)-\delta(f+f_c)\exp(-j\theta)]$
$\text{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j\text{sgn}(f)$
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)=f_0\sum_{n=-\infty}^{\infty}\delta\left(f-nf_0\right)$

Time domain $x(t)$	Frequency domain $X(f)$	Property
$g(t-a)$	$\exp(-j2\pi fa)G(f)$	Time shifting
$\exp(-j2\pi f_c t)g(t)$	$G(f-f_c)$	Frequency shifting
$g(bt)$	$\frac{G(f/b)}{\|b\|}$	Time scaling
$g(bt-a)$	$\frac{1}{\|b\|}\exp(-j2\pi a(f/b))\cdot G(f/b)$	Time scaling and shifting
$\frac{d}{dt}g(t)$	$j2\pi fG(f)$	Differentiation wrt time
$tg(t)$	$\frac{1}{2\pi}\frac{d}{df}G(f)$	Differentiation wrt frequency
$g^*(t)$	$G^*(-f)$	Conjugate functions
$G(t)$	$g(-f)$	Duality
$\int_{-\infty}^{t}g(\tau)d\tau$	$\frac{1}{j2\pi f}G(f)+\frac{G(0)}{2}\delta(f)$	Integration wrt time
$g(t)h(t)$	$G(f)*H(f)$	Time multiplication
$g(t)*h(t)$	$G(f)H(f)$	Time convolution
$ag(t)+bh(t)$	$aG(f)+bH(f)$	Linearity a, b constants
$\int_{-\infty}^{\infty}x(t)y^*(t)dt$	$\int_{-\infty}^{\infty}X(f)Y^*(f)df$	Parseval's theorem
$E_x=\int_{-\infty}^{\infty}\|x(t)\|^2dt$	$E_x=\int_{-\infty}^{\infty}\|X(f)\|^2df$	Parseval's theorem

Description	Property
$g(0)=\int_{-\infty}^{\infty}G(f)df$	Area under $G(f)$
$G(0)=\int_{-\infty}^{\infty}G(t)dt$	Area under $g(t)$

$$
\begin{aligned}
u(t) &= \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases} && \text{Unit Step Function} \\
\operatorname{sgn}(t) &= \begin{cases} +1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} && \text{Signum Function} \\
\operatorname{sinc}(2Wt) &= \frac{\sin(2\pi Wt)}{2\pi Wt} && \text{sinc Function} \\
\operatorname{rect}(t) = \Pi(t) &= \begin{cases} 1, & -0.5 < t < 0.5 \\ 0, & |t| > 0.5 \end{cases} && \text{Rectangular/Gate Function} \\
\operatorname{tri}(t/T) &= \begin{cases} 1 - \frac{|t|}{T}, & |t| < T \\ 0, & |t| \geq T \end{cases} = \frac{1}{T}\Pi(t/T) * \Pi(t/T) && \text{Triangle Function} \\
g(t) * h(t) = (g * h)(t) &= \int_{\infty}^{\infty} g(\tau)h(t-\tau)d\tau && \text{Convolution}
\end{aligned}
$$

Fourier transform of continuous time periodic signal

Required for some questions on **sampling**:

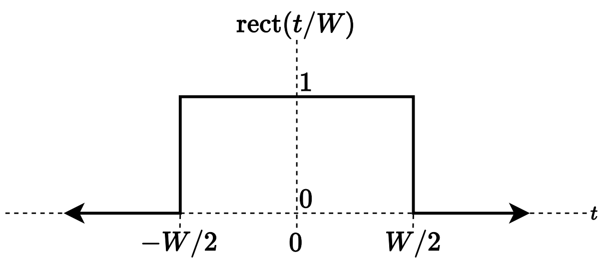
Transform a continuous time-periodic signal $x_p(t) = \sum_{n=-\infty}^{\infty} x(t - nT_s)$ with period T_s:

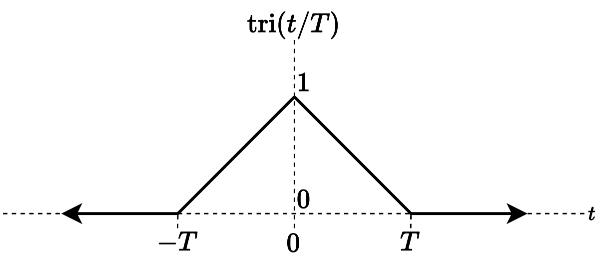
$$
X_p(f) = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_s) \quad f_s = \frac{1}{T_s}
$$

Calculate C_n coefficient as follows from $x_p(t)$:

$$
C_n = \frac{1}{T_s} \int_{T_s} x_p(t) \exp(-j2\pi f_s t) dt = \frac{1}{T_s} X(nf_s) \quad \text{(TODO: Check)} \quad x(t - nT_s) \text{ is contained in the interval } T_s
$$

Shape functions





Random processes examples

$$
\begin{aligned}
&\text{Example: separate RV from expression} \\
&X(t) = A\cos(2\pi f_c t) \quad A \sim \mathcal{N}(\mu = 5, \sigma^2 = 1) \\
\implies &E[X(t)] = E[A\cos(2\pi f_c t)] = E[A]\cos(2\pi f_c t) = 5\cos(2\pi f_c t) \\
&\text{Example: random phase} \\
&X(t) = B\cos(2\pi f_c t + \theta) \quad \theta \sim \mathcal{U}(0, 2\pi) \\
\implies &E[X(t)] = E[B\cos(2\pi f_c t + \theta)] = B \int_0^{2\pi} \underbrace{\frac{1}{2\pi}}_{\text{uniform}} \cos(2\pi f_c t + \theta) d\theta = 0
\end{aligned}
$$

Wide sense stationary (WSS)

Two conditions for WSS:

Constant mean	**Autocorrelation only dependent on time difference**
$\mu_X(t) = \mu_X$ Constant	$R_{XX}(t_1, t_2) = R_X(t_1 - t_2) = R_X(\tau)$
$\mu_X(t) = E[X(t)]$	$E[X(t_1)X(t_2)] = E[X(t)X(t+\tau)]$

Ergodicity

$$\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} x(t)dt$$
$$\langle X(t+\tau)X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} x(t+\tau)x(t)dt$$
$$E[\langle X(t)\rangle_T] = \frac{1}{2T}\int_{-T}^{T} x(t)dt = \frac{1}{2T}\int_{-T}^{T} m_X dt = m_X$$

Type	Normal	Mean square sense
ergodic in mean	$\lim_{T\to\infty}\langle X(t)\rangle_T = m_X(t) = m_X$	$\lim_{T\to\infty}\mathrm{VAR}[\langle X(t)\rangle_T] = 0$
ergodic in autocorrelation function	$\lim_{T\to\infty}\langle X(t+\tau)X(t)\rangle_T = R_X(\tau)$	$\lim_{T\to\infty}\mathrm{VAR}[\langle X(t+\tau)X(t)\rangle_T] = 0$

Note: **A WSS random process needs to be both ergodic in mean and autocorrelation to be considered an ergodic process**

Other identities

$$f*(g*h) = (f*g)*h \quad \text{Convolution associative}$$
$$a(f*g) = (af)*g \quad \text{Convolution associative}$$
$$\sum_{x=-\infty}^{\infty}(f(xa)\delta(\omega - xb)) = f\left(\frac{\omega a}{b}\right)$$

Other trig

$$\cos 2\theta = 2\cos^2\theta - 1 \Leftrightarrow \frac{\cos 2\theta + 1}{2} = \cos^2\theta$$
$$e^{-j\alpha} - e^{j\alpha} = -2j\sin(\alpha)$$
$$e^{-j\alpha} + e^{j\alpha} = 2\cos(\alpha)$$
$$\cos(-A) = \cos(A)$$
$$\sin(-A) = -\sin(A)$$
$$\sin(A+\pi/2) = \cos(A)$$
$$\sin(A-\pi/2) = -\cos(A)$$
$$\cos(A-\pi/2) = \sin(A)$$
$$\cos(A+\pi/2) = -\sin(A)$$
$$\int_{x\in\mathbb{R}} \mathrm{sinc}(Ax) = \frac{1}{|A|}$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$
$$\cos(A)\sin(B) = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$
$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A}{2} - \frac{B}{2}\right)\cos\left(\frac{A}{2} + \frac{B}{2}\right)$$
$$\cos(A) - \cos(B) = -2\sin\left(\frac{A}{2} - \frac{B}{2}\right)\sin\left(\frac{A}{2} + \frac{B}{2}\right)$$
$$\sin(A) + \sin(B) = 2\sin\left(\frac{A}{2} + \frac{B}{2}\right)\cos\left(\frac{A}{2} - \frac{B}{2}\right)$$
$$\sin(A) - \sin(B) = 2\sin\left(\frac{A}{2} - \frac{B}{2}\right)\cos\left(\frac{A}{2} + \frac{B}{2}\right)$$
$$\cos(A) + \sin(B) = -2\sin\left(\frac{A}{2} - \frac{B}{2} - \frac{\pi}{4}\right)\sin\left(\frac{A}{2} + \frac{B}{2} + \frac{\pi}{4}\right)$$
$$\cos(A) - \sin(B) = -2\sin\left(\frac{A}{2} + \frac{B}{2} - \frac{\pi}{4}\right)\sin\left(\frac{A}{2} - \frac{B}{2} + \frac{\pi}{4}\right)$$

IQ/Complex envelope

Def. $\tilde{g}(t) = g_I(t) + jg_Q(t)$ as the complex envelope. Best to convert to $e^{j\theta}$ form.

Convert complex envelope representation to time-domain representation of signal

$$
\begin{aligned}
g(t) &= g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t) \\
&= \text{Re}[\tilde{g}(t)\exp(j2\pi f_c t)] \\
&= A(t)\cos(2\pi f_c t + \phi(t)) \\
A(t) &= |g(t)| = \sqrt{g_I^2(t) + g_Q^2(t)} \quad \text{Amplitude} \\
\phi(t) & \quad \text{Phase} \\
g_I(t) &= A(t)\cos(\phi(t)) \quad \text{In-phase component} \\
g_Q(t) &= A(t)\sin(\phi(t)) \quad \text{Quadrature-phase component}
\end{aligned}
$$

For transfer function

$$
\begin{aligned}
h(t) &= h_I(t)\cos(2\pi f_c t) - h_Q(t)\sin(2\pi f_c t) \\
&= 2\text{Re}[\tilde{h}(t)\exp(j2\pi f_c t)] \\
\Rightarrow \tilde{h}(t) &= h_I(t)/2 + jh_Q(t)/2 = A(t)/2\exp(j\phi(t))
\end{aligned}
$$

AM

Conventional AM modulation (CAM)

$$
\begin{aligned}
x(t) &= A_c\cos(2\pi f_c t)\,[1 + k_a m(t)] = A_c\cos(2\pi f_c t)\,[1 + m_a m(t)/A_c] \quad \text{CAM signal} \\
& \text{where } m(t) = A_m\hat{m}(t) \text{ and } \hat{m}(t) \text{ is the normalized modulating signal} \\
m_a &= \frac{|\min_t(k_a m(t))|}{A_c} \quad k_a \text{ is the amplitude sensitivity (volt}^{-1}\text{)}, m_a \text{ is the modulation index.} \\
m_a &= \frac{A_{\max} - A_{\min}}{A_{\max} + A_{\min}} \quad \text{(Symmetrical } m(t)\text{)} \\
m_a &= k_a A_m \quad \text{(Symmetrical } m(t)\text{)} \\
P_c &= \frac{{A_c}^2}{2} \quad \text{Carrier power} \\
P_s &= \frac{1}{4}{m_a}^2{A_c}^2 \quad \text{Signal power, \textbf{total} of all 4 sideband power, \textbf{single-tone} case} \\
\eta &= \frac{\text{Signal Power}}{\text{Total Power}} = \frac{P_s}{P_s + P_c} = \frac{P_s}{P_x} \quad \text{Power efficiency} \\
B_T &= 2f_m = 2B
\end{aligned}
$$

B_T: Signal bandwidth B: Bandwidth of modulating wave

Overmodulation (resulting in phase reversals at crossing points): $m_a > 1$

Double sideband suppressed carrier (DSB-SC)

$$
\begin{aligned}
x_{\text{DSB}}(t) &= A_c\cos(2\pi f_c t)m(t) \\
B_T &= 2f_m = 2B
\end{aligned}
$$

FM/PM

$$
\begin{aligned}
s(t) &= A_c\cos[2\pi f_c t + k_p m(t)] \quad \text{Phase modulated (PM)} \\
s(t) &= A_c\cos(\theta_i(t)) = A_c\cos\left[2\pi f_c t + 2\pi k_f\int_{-\infty}^{t} m(\tau)d\tau\right] \quad \text{Frequency modulated (FM)} \\
s(t) &= A_c\cos[2\pi f_c t + \beta\sin(2\pi f_m t)] \quad \text{FM single tone} \\
f_i(t) &= \frac{1}{2\pi}\frac{d}{dt}\theta_i(t) = f_c + k_f m(t) = f_c + \Delta f_{\max}\hat{m}(t) \quad \text{Instantaneous frequency} \\
\Delta f_{\max} &= \max_t|f_i(t) - f_c| = k_f\max_t|m(t)| \quad \text{Maximum frequency deviation} \\
\Delta f_{\max} &= k_f A_m \quad \text{Maximum frequency deviation (sinusoidal)} \\
\beta &= \frac{\Delta f_{\max}}{f_m} \quad \text{Modulation index} \\
D &= \frac{\Delta f_{\max}}{W_m} \quad \text{Deviation ratio, where } W_m \text{ is bandwidth of } m(t) \text{ (Use FT)}
\end{aligned}
$$

Bessel function

$$J_n(\beta) = \begin{cases} J_{-n}(\beta) & n \text{ is even} \\ -J_{-n}(\beta) & n \text{ is odd} \end{cases}$$
$$1 = \sum_{n\in\mathbb{Z}} {J_n}^2(\beta) \qquad \text{Conservation of power}$$

Bessel form of FM signal

$$s(t) = A_c \cos\left[2\pi f_c t + \beta \sin(2\pi f_m t)\right]$$
$$\iff s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c + n f_m)t]$$

FM signal power

$$P_{\text{av}} = \frac{{A_c}^2}{2} \qquad \text{Av. power of full signal}$$
$$P_{\text{i}} = \frac{{A_c}^2 |J_{\text{i}}(\beta)|^2}{2} \qquad \text{Av. power of band } i$$
$$i = 0 \implies f_c + 0 f_m \qquad \text{Middle band}$$
$$i = 1 \implies f_c + 1 f_m \qquad \text{1st sideband}$$
$$i = -1 \implies f_c - 1 f_m \qquad \text{-1st sideband}$$
$$\cdots$$

Carson's rule to find B (98% power bandwidth rule)

$$B = 2(\beta + 1) f_m$$
$$B = 2(\Delta f_{\max} + f_m)$$
$$B = 2(D + 1) W_m$$
$$B = \begin{cases} 2(\Delta f_{\max} + f_m) = 2(\Delta f_{\max} + W_m) & \text{FM, sinusoidal message} \\ 2(\Delta \phi_{\max} + 1) f_m = 2(\Delta \phi_{\max} + 1) W_m & \text{PM, sinusoidal message} \end{cases}$$

$$D < 1, \beta < 1 \implies \text{Narrowband} \quad D > 1, \beta > 1 \implies \text{Wideband}$$

Complex envelope of a FM signal

$$s(t) = A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t))$$
$$\iff \tilde{s}(t) = A_c \exp(j\beta \sin(2\pi f_m t))$$
$$s(t) = \text{Re}[\tilde{s}(t) \exp\left(j 2\pi f_c t\right)]$$
$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j 2\pi f_m t)$$

Power, energy and autocorrelation

$$G_{\text{WGN}}(f) = \frac{N_0}{2}$$
$$G_x(f) = |H(f)|^2 G_w(f) \text{ (PSD)}$$
$$G_x(f) = G(f) G_w(f) \text{ (PSD)}$$
$$G_x(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} \text{ (PSD)}$$
$$G_x(f) = \mathfrak{F}[R_x(\tau)] \text{ (WSS)}$$
$$P_x = {\sigma_x}^2 = \int_{\mathbb{R}} G_x(f) df \quad \text{For zero mean}$$
$$P_x = {\sigma_x}^2 = \lim_{t\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \quad \text{For zero mean}$$
$$P[A \cos(2\pi f t + \phi)] = \frac{A^2}{2} \quad \text{Power of sinusoid}$$
$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \quad \text{Parseval's theorem}$$
$$R_x(\tau) = \mathfrak{F}(G_x(f)) \quad \text{PSD to Autocorrelation}$$
$$P_x = R_x(0) \quad \text{Average power of WSS process } x(t)$$

White noise

$$
\begin{aligned}
R_W(\tau) &= \frac{N_0}{2}\delta(\tau) = \frac{kT}{2}\delta(\tau) = \sigma^2\delta(\tau) \\
G_w(f) &= \frac{N_0}{2}
\end{aligned}
$$

Noise performance

Use formualas from previous section, Power, energy and autocorrelation.
Use these formulas in particular:

$$
\begin{aligned}
G_{\text{WGN}}(f) &= \frac{N_0}{2} \\
G_x(f) &= |H(f)|^2 G_w(f) && \text{Note the square in } |H(f)|^2 \\
P_x = {\sigma_x}^2 &= \int_{\mathbb{R}} G_x(f)df && \text{Often perform graphical integration}
\end{aligned}
$$

$$
\begin{aligned}
\text{CNR}_{\text{in}} &= \frac{P_{\text{in}}}{P_{\text{noise}}} \\
\text{CNR}_{\text{in,FM}} &= \frac{A^2}{2WN_0} \\
\text{SNR}_{\text{FM}} &= \frac{3A^2k_f^2P}{2N_0W^3} \\
\text{SNR(dB)} &= 10\log_{10}(\text{SNR}) \quad \text{Decibels from ratio}
\end{aligned}
$$

Sampling

$$
\begin{aligned}
t &= nT_s \\
T_s &= \frac{1}{f_s} \\
x_s(t) &= x(t)\delta_s(t) = x(t)\sum_{n\in\mathbb{Z}}\delta(t-nT_s) = \sum_{n\in\mathbb{Z}} x(nT_s)\delta(t-nT_s) \\
X_s(f) &= f_sX(f) * \sum_{n\in\mathbb{Z}}\delta\left(f-\frac{n}{T_s}\right) = f_sX(f) * \sum_{n\in\mathbb{Z}}\delta\left(f-nf_s\right) \\
\implies X_s(f) &= \sum_{n\in\mathbb{Z}} f_sX\left(f-nf_s\right) \quad \text{Sampling (FT)} \\
B &> \frac{1}{2}f_s \implies 2B > f_s \to \text{Aliasing}
\end{aligned}
$$

Procedure to reconstruct sampled signal

Analog signal $x'(t)$ which can be reconstructed from a sampled signal $x_s(t)$: Put $x_s(t)$ through LPF with maximum frequency of $f_s/2$ and minimum frequency of $-f_s/2$. Anything outside of the BPF will be attenuated, therefore n which results in frequencies outside the BPF will evaluate to 0 and can be ignored.

Example: $f_s = 5000 \implies \text{LPF} \in [-2500, 2500]$

Then iterate for $n = 0, 1, -1, 2, -2, \ldots$ until the first iteration where the result is 0 since all terms are eliminated by the LPF.

TODO: Add example

Then add all terms and transform $\bar{X}_s(f)$ back to time domain to get $x_s(t)$

Fourier transform of continuous time periodic signal (1)

Required for some questions on **sampling**:

Transform a continuous time-periodic signal $x_p(t) = \sum_{n=-\infty}^{\infty} x(t-nT_s)$ with period T_s:

$$
X_p(f) = \sum_{n=-\infty}^{\infty} C_n\delta(f-nf_s) \quad f_s = \frac{1}{T_s}
$$

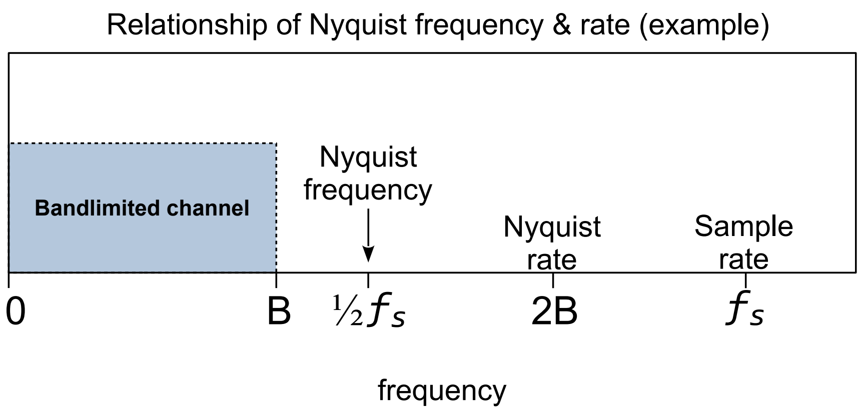
Calculate C_n coefficient as follows from $x_p(t)$:

$$
\begin{aligned}
C_n &= \frac{1}{T_s}\int_{T_s} x_p(t)\exp(-j2\pi f_st)dt \\
&= \frac{1}{T_s}X(nf_s) \quad \text{(TODO: Check)} \quad x(t-nT_s) \text{ is contained in the interval } T_s
\end{aligned}
$$

Nyquist criterion for zero-ISI

Do not transmit more than $2B$ samples per second over a channel of B bandwidth.

$$\text{Nyquist rate} = 2B \quad \text{Nyquist interval} = \frac{1}{2B}$$



Insert here figure 8.3 from M F Mesiya - Contemporary Communication Systems (Add image to `images/sampling.png`)

Cannot add directly due to copyright! **TODO: Make an open source replacement for this diagram** **Send a PR to GitHub.**

sampling

Quantizer

$$\Delta = \frac{x_{\text{Max}} - x_{\text{Min}}}{2^k} \quad \text{for } k\text{-bit quantizer (V/lsb)} \quad \text{Quantizer step size } \Delta$$

Quantization noise

$$\begin{aligned} e &:= y - x \quad \text{Quantization error} \\ \mu_E &= E[E] = 0 \quad \text{Zero mean} \\ P_E &= {\sigma_E}^2 = \frac{\Delta^2}{12} = 2^{-2m} V^2/3 \quad \text{Uniformly distributed error} \\ \text{SQNR} &= \frac{\text{Signal power}}{\text{Quantization noise power}} = \frac{P_x}{P_E} \\ \text{SQNR(dB)} &= 10 \log_{10}(\text{SQNR}) \\ m \to m + A \text{ bits} &\implies \text{newSQNR(dB)} = \text{SQNR(dB)} + 6A \text{ dB} \end{aligned}$$

Insert here figure 8.17 from M F Mesiya - Contemporary Communication Systems (Add image to `images/quantizer.png`)

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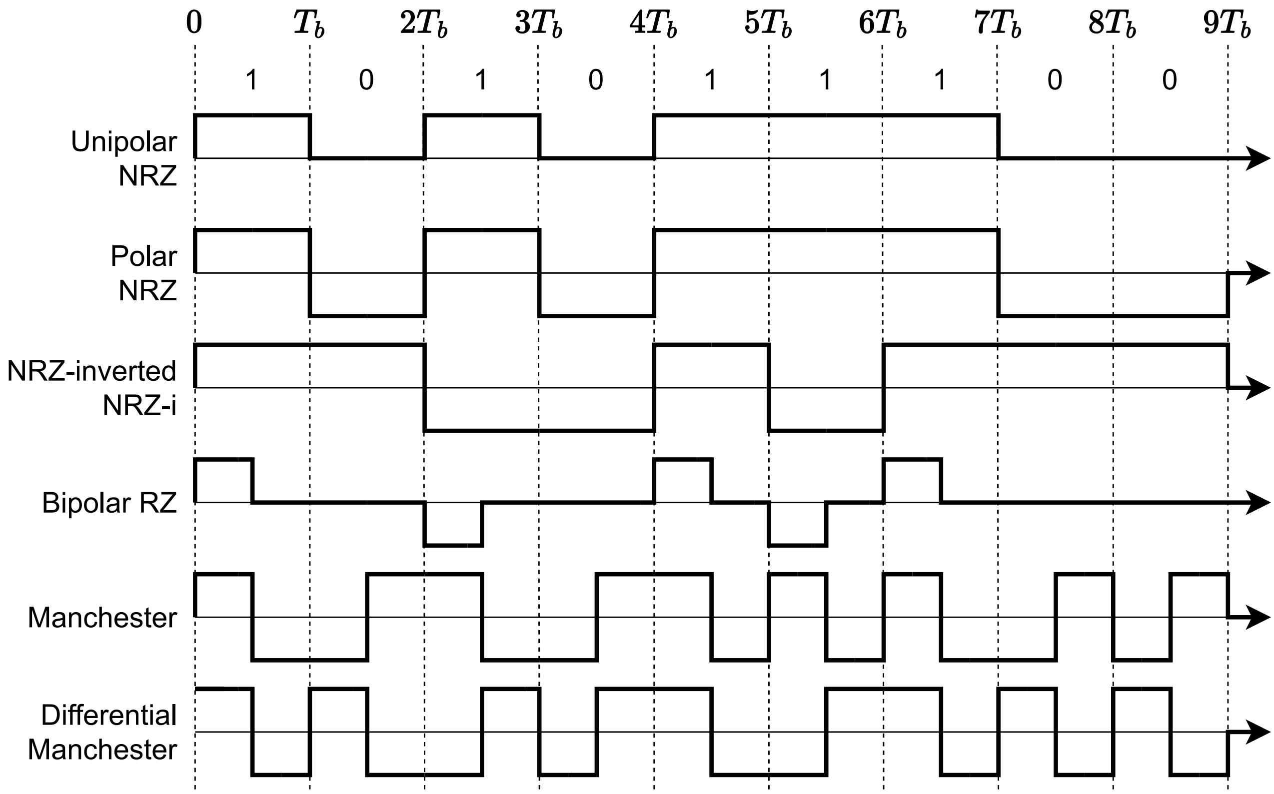
quantizer

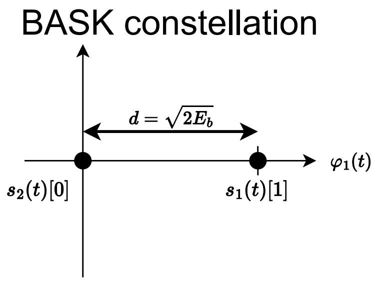
Line codes

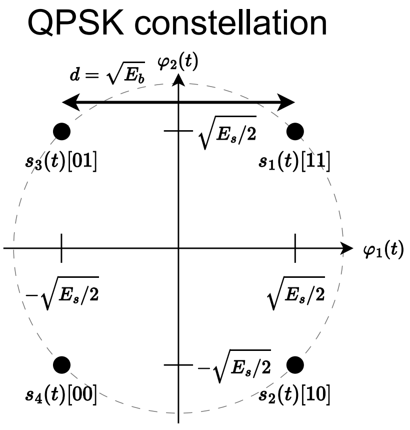


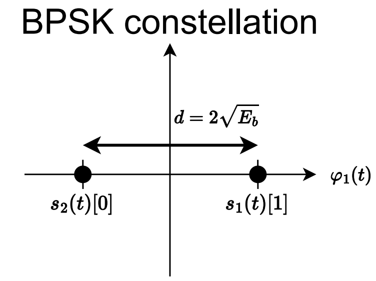
$$R_b \to \text{Bit rate}$$
$$D \to \text{Symbol rate} \mid R_d \mid 1/T_b$$
$$A \to m_a$$
$$V(f) \to \text{Pulse shape}$$
$$V_{\text{rectangle}}(f) = T\text{sinc}(fT \times \text{DutyCycle})$$
$$G_{\text{MunipolarNRZ}}(f) = \frac{(M^2-1)A^2D}{12}|V(f)|^2 + \frac{(M-1)^2}{4}(DA)^2\sum_{l=-\infty}^{\infty}|V(lD)|^2\delta(f-lD)$$
$$G_{\text{MpolarNRZ}}(f) = \frac{(M^2-1)A^2D}{3}|V(f)|^2$$
$$G_{\text{unipolarNRZ}}(f) = \frac{A^2}{4R_b}\left(\text{sinc}^2\left(\frac{f}{R_b}\right) + R_b\delta(f)\right), \text{NB}_0 = R_b$$
$$G_{\text{polarNRZ}}(f) = \frac{A^2}{R_b}\text{sinc}^2\left(\frac{f}{R_b}\right)$$
$$G_{\text{unipolarNRZ}}(f) = \frac{A^2}{4R_b}\left(\text{sinc}^2\left(\frac{f}{R_b}\right) + R_b\delta(f)\right)$$
$$G_{\text{unipolarRZ}}(f) = \frac{A^2}{16}\left(\sum_{l=-\infty}^{\infty}\delta\left(f-\frac{l}{T_b}\right)|\text{sinc}(\text{duty}\times l)|^2 + T_b\,|\text{sinc}\,(\text{duty}\times fT_b)|^2\right), \text{NB}_0 = 2R_b$$

TODO: Someone please make plots of the PSD for all line code types in Mathematica or Python! Send a PR to GitHub.

Modulation and basis functions







BASK

Basis functions

$$\varphi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t) \quad 0 \leq t \leq T_b$$

Symbol mapping

$$b_n : \{1, 0\} \to a_n : \{1, 0\}$$

2 possible waveforms

$$s_1(t) = A_c\sqrt{\frac{T_b}{2}}\varphi_1(t) = \sqrt{2E_b}\varphi_1(t)$$
$$s_1(t) = 0$$
$$\text{Since } E_b = E_{\text{average}} = \tfrac{1}{2}(\tfrac{A_c{}^2}{2} \times T_b + 0) = \tfrac{A_c{}^2}{4}T_b$$

Distance is $d = \sqrt{2E_b}$

BPSK

Basis functions

$$\varphi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t) \quad 0 \leq t \leq T_b$$

Symbol mapping

$$b_n : \{1, 0\} \to a_n : \{1, -1\}$$

2 possible waveforms

$$s_1(t) = A_c\sqrt{\frac{T_b}{2}}\varphi_1(t) = \sqrt{E_b}\varphi_1(t)$$
$$s_1(t) = -A_c\sqrt{\frac{T_b}{2}}\varphi_1(t) = -\sqrt{E_b}\varphi_2(t)$$
$$\text{Since } E_b = E_{\text{average}} = \tfrac{1}{2}(\tfrac{A_c{}^2}{2} \times T_b + \tfrac{A_c{}^2}{2} \times T_b) = \tfrac{A_c{}^2}{2}T_b$$

Distance is $d = 2\sqrt{E_b}$

QPSK ($M = 4$ PSK)

Basis functions

$$T = 2T_b \quad \text{Time per symbol for two bits } T_b$$

$$\varphi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \quad 0 \le t \le T$$

$$\varphi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \quad 0 \le t \le T$$

4 possible waveforms

$$s_1(t) = \sqrt{E_s/2}\,[\varphi_1(t) + \varphi_2(t)]$$
$$s_2(t) = \sqrt{E_s/2}\,[\varphi_1(t) - \varphi_2(t)]$$
$$s_3(t) = \sqrt{E_s/2}\,[-\varphi_1(t) + \varphi_2(t)]$$
$$s_4(t) = \sqrt{E_s/2}\,[-\varphi_1(t) - \varphi_2(t)]$$

Note on energy per symbol: Since $|s_i(t)| = A_c$, have to normalize distance as follows:

$$\begin{aligned} s_i(t) &= A_c\sqrt{T/2}/\sqrt{2} \times [\alpha_{1i}\varphi_1(t) + \alpha_{2i}\varphi_2(t)] \\ &= \sqrt{T{A_c}^2/4}\,[\alpha_{1i}\varphi_1(t) + \alpha_{2i}\varphi_2(t)] \\ &= \sqrt{E_s/2}\,[\alpha_{1i}\varphi_1(t) + \alpha_{2i}\varphi_2(t)] \end{aligned}$$

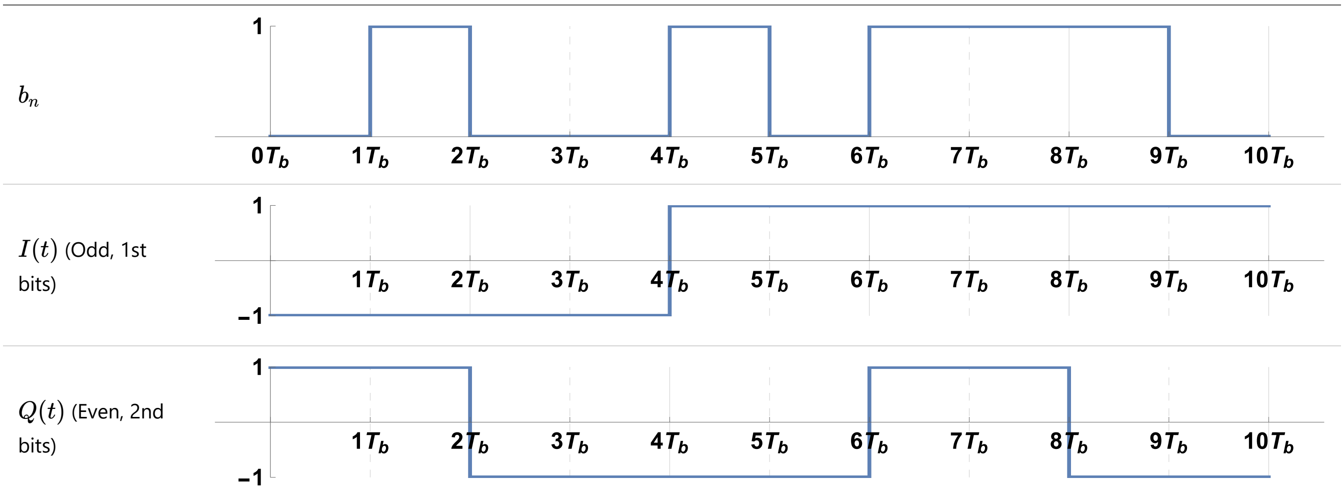
Signal

$$\text{Symbol mapping: } \{1, 0\} \to \{1, -1\}$$
$$I(t) = b_{2n}\varphi_1(t) \quad \text{Even bits}$$
$$Q(t) = b_{2n+1}\varphi_2(t) \quad \text{Odd bits}$$
$$x(t) = A_c[I(t)\cos(2\pi f_c t) - Q(t)\sin(2\pi f_c t)]$$

Example of waveform

▸ Code

Remember that $T = 2T_b$



Matched filter

1. Filter function

Find transfer function $h(t)$ of matched filter and apply to an input:
Note that $x(T-t)$ is equivalent to horizontally flipping $x(t)$ around $x = T/2$.

$$
\begin{aligned}
h(t) &= s_1(T-t) - s_2(T-t) \\
h(t) &= s^*(T-t) \qquad ((.)^* \text{ is the conjugate}) \\
s_{on}(t) &= h(t) * s_n(t) = \int_{\infty}^{\infty} h(\tau) s_n(t-\tau) d\tau \quad \text{Filter output} \\
n_o(t) &= h(t) * n(t) \quad \text{Noise at filter output}
\end{aligned}
$$

2. Bit error rate of matched filter

Bit error rate (BER) from matched filter outputs and filter output noise

$$
\begin{aligned}
Q(x) &= \frac{1}{2} - \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{2}}\right) \Leftrightarrow \text{erf}\left(\frac{x}{\sqrt{2}}\right) = 1 - 2Q(x) \\
E_b &= d^2 = \int_{-\infty}^{\infty} |s_1(t) - s_2(t)|^2 dt \quad \text{Energy per bit/Distance} \\
T &= 1/R_b \quad R_b\text{: Bitrate} \\
E_b &= P_{\text{av}} T = P_{\text{av}}/R_b \quad \text{Energy per bit} \\
P_{\text{av}} &= E_b/T = E_b R_b \quad \text{Average power} \\
P(\text{W}) &= 10^{\frac{P(\text{dB})}{10}} \\
P_{\text{RX}}(W) &= P_{\text{TX}}(W) \cdot 10^{\frac{P_{\text{loss}}(\text{dB})}{10}} \quad P_{\text{loss}} \text{ is expressed with negative sign e.g. "-130 dB"} \\
\text{BER}_{\text{MatchedFilter}} &= Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{2N_0}}\right) \\
\text{BER}_{\text{unipolarNRZ|BASK}} &= Q\left(\sqrt{\frac{d^2}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right) \\
\text{BER}_{\text{polarNRZ|BPSK}} &= Q\left(\sqrt{\frac{2d^2}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)
\end{aligned}
$$

Value tables for $\mathrm{erf}(x)$ and $Q(x)$

$Q(x)$ function

You should use erf function table instead in exams using the identity $Q(x) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)$. Use this for validation.

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0.00	0.5	2.30	0.010724	4.55	2.6823×10^{-6}	6.80	5.231×10^{-12}
0.05	0.48006	2.35	0.0093867	4.60	2.1125×10^{-6}	6.85	3.6925×10^{-12}
0.10	0.46017	2.40	0.0081975	4.65	1.6597×10^{-6}	6.90	2.6001×10^{-12}
0.15	0.44038	2.45	0.0071428	4.70	1.3008×10^{-6}	6.95	1.8264×10^{-12}
0.20	0.42074	2.50	0.0062097	4.75	1.0171×10^{-6}	7.00	1.2798×10^{-12}
0.25	0.40129	2.55	0.0053861	4.80	7.9333×10^{-7}	7.05	8.9459×10^{-13}
0.30	0.38209	2.60	0.0046612	4.85	6.1731×10^{-7}	7.10	6.2378×10^{-13}
0.35	0.36317	2.65	0.0040246	4.90	4.7918×10^{-7}	7.15	4.3389×10^{-13}
0.40	0.34458	2.70	0.003467	4.95	3.7107×10^{-7}	7.20	3.0106×10^{-13}
0.45	0.32636	2.75	0.0029798	5.00	2.8665×10^{-7}	7.25	2.0839×10^{-13}
0.50	0.30854	2.80	0.0025551	5.05	2.2091×10^{-7}	7.30	1.4388×10^{-13}
0.55	0.29116	2.85	0.002186	5.10	1.6983×10^{-7}	7.35	9.9103×10^{-14}
0.60	0.27425	2.90	0.0018658	5.15	1.3024×10^{-7}	7.40	6.8092×10^{-14}
0.65	0.25785	2.95	0.0015889	5.20	9.9644×10^{-8}	7.45	4.667×10^{-14}
0.70	0.24196	3.00	0.0013499	5.25	7.605×10^{-8}	7.50	3.1909×10^{-14}
0.75	0.22663	3.05	0.0011442	5.30	5.7901×10^{-8}	7.55	2.1763×10^{-14}
0.80	0.21186	3.10	0.0009676	5.35	4.3977×10^{-8}	7.60	1.4807×10^{-14}
0.85	0.19766	3.15	0.00081635	5.40	3.332×10^{-8}	7.65	1.0049×10^{-14}
0.90	0.18406	3.20	0.00068714	5.45	2.5185×10^{-8}	7.70	6.8033×10^{-15}
0.95	0.17106	3.25	0.00057703	5.50	1.899×10^{-8}	7.75	4.5946×10^{-15}
1.00	0.15866	3.30	0.00048342	5.55	1.4283×10^{-8}	7.80	3.0954×10^{-15}
1.05	0.14686	3.35	0.00040406	5.60	1.0718×10^{-8}	7.85	2.0802×10^{-15}
1.10	0.13567	3.40	0.00033693	5.65	8.0224×10^{-9}	7.90	1.3945×10^{-15}
1.15	0.12507	3.45	0.00028029	5.70	5.9904×10^{-3}	7.95	9.3256×10^{-16}
1.20	0.11507	3.50	0.00023263	5.75	4.4622×10^{-9}	8.00	6.221×10^{-16}
1.25	0.10565	3.55	0.00019262	5.80	3.3157×10^{-9}	8.05	4.1397×10^{-16}
1.30	0.0968	3.60	0.00015911	5.85	2.4579×10^{-9}	8.10	2.748×10^{-16}
1.35	0.088508	3.65	0.00013112	5.90	1.8175×10^{-9}	8.15	1.8196×10^{-16}
1.40	0.080757	3.70	0.0001078	5.95	1.3407×10^{-9}	8.20	1.2019×10^{-16}
1.45	0.073529	3.75	8.8417×10^{-5}	6.00	9.8659×10^{-10}	8.25	7.9197×10^{-17}
1.50	0.066807	3.80	7.2348×10^{-5}	6.05	7.2423×10^{-10}	8.30	5.2056×10^{-17}
1.55	0.060571	3.85	5.9059×10^{-5}	6.10	5.3034×10^{-10}	8.35	3.4131×10^{-17}
1.60	0.054799	3.90	4.8096×10^{-5}	6.15	3.8741×10^{-10}	8.40	2.2324×10^{-17}
1.65	0.049471	3.95	3.9076×10^{-5}	6.20	2.8232×10^{-10}	8.45	1.4565×10^{-17}
1.70	0.044565	4.00	3.1671×10^{-5}	6.25	2.0523×10^{-10}	8.50	9.4795×10^{-18}
1.75	0.040059	4.05	2.5609×10^{-5}	6.30	1.4882×10^{-10}	8.55	6.1544×10^{-18}
1.80	0.03593	4.10	2.0658×10^{-5}	6.35	1.0766×10^{-10}	8.60	3.9858×10^{-18}
1.85	0.032157	4.15	1.6624×10^{-5}	6.40	7.7688×10^{-11}	8.65	2.575×10^{-18}
1.90	0.028717	4.20	1.3346×10^{-5}	6.45	5.5925×10^{-11}	8.70	1.6594×10^{-18}

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
1.95	0.025588	4.25	1.0689×10^{-5}	6.50	4.016×10^{-11}	8.75	1.0668×10^{-18}
2.00	0.02275	4.30	8.5399×10^{-6}	6.55	2.8769×10^{-11}	8.80	6.8408×10^{-19}
2.05	0.020182	4.35	6.8069×10^{-6}	6.60	2.0558×10^{-11}	8.85	4.376×10^{-19}
2.10	0.017864	4.40	5.4125×10^{-6}	6.65	1.4655×10^{-11}	8.90	2.7923×10^{-19}
2.15	0.015778	4.45	4.2935×10^{-6}	6.70	1.0421×10^{-11}	8.95	1.7774×10^{-19}
2.20	0.013903	4.50	3.3977×10^{-6}	6.75	7.3923×10^{-12}	9.00	1.1286×10^{-19}
2.25	0.012224						

Adapted from table 6.1 M F Mesiya - Contemporary Communication Systems

$\mathrm{erf}(x)$ function

$$Q(x) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}(\frac{x}{\sqrt{2}})$$

x	$\mathrm{erf}(x)$	x	$\mathrm{erf}(x)$	x	$\mathrm{erf}(x)$
0.00	0.00000	0.75	0.71116	1.50	0.96611
0.05	0.05637	0.80	0.74210	1.55	0.97162
0.10	0.11246	0.85	0.77067	1.60	0.97635
0.15	0.16800	0.90	0.79691	1.65	0.98038
0.20	0.22270	0.95	0.82089	1.70	0.98379
0.25	0.27633	1.00	0.84270	1.75	0.98667
0.30	0.32863	1.05	0.86244	1.80	0.98909
0.35	0.37938	1.10	0.88021	1.85	0.99111
0.40	0.42839	1.15	0.89612	1.90	0.99279
0.45	0.47548	1.20	0.91031	1.95	0.99418
0.50	0.52050	1.25	0.92290	2.00	0.99532
0.55	0.56332	1.30	0.93401	2.50	0.99959
0.60	0.60386	1.35	0.94376	3.00	0.99998
0.65	0.64203	1.40	0.95229	3.30	0.999998**
0.70	0.67780	1.45	0.95970		

**The value of $\mathrm{erf}(3.30)$ should be ≈ 0.999997 instead, but this value is quoted in the formula table.

$Q(x)$ fast reference

Using identity.

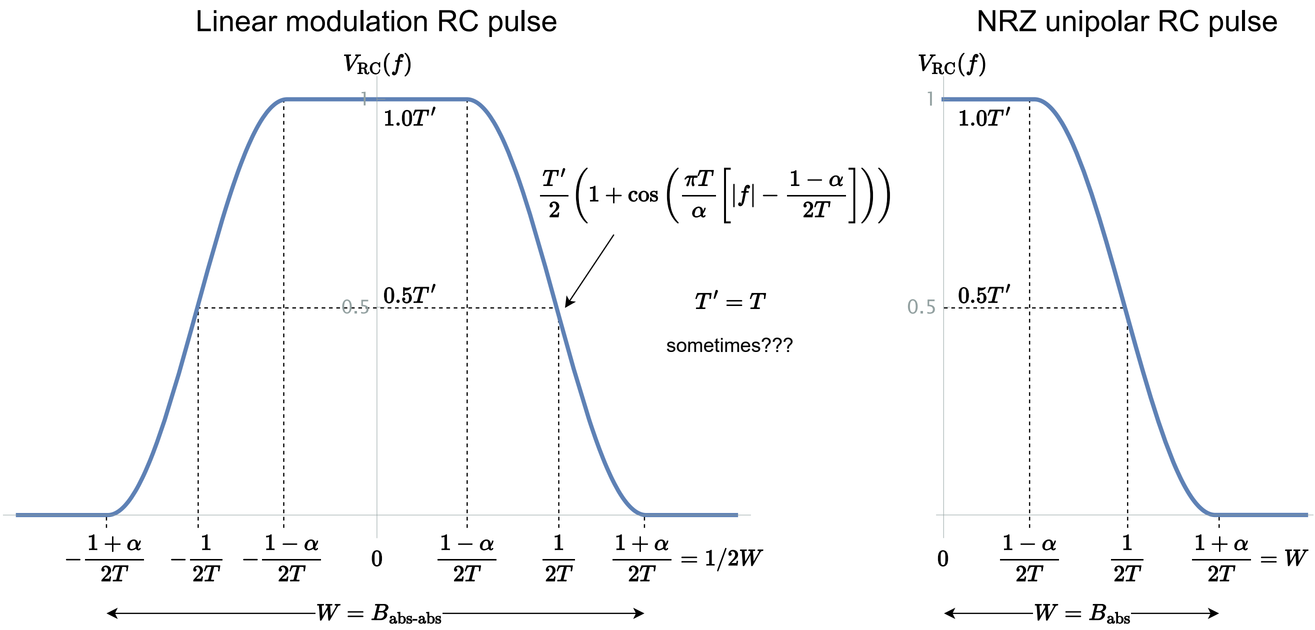
x	$Q(x)$
$\sqrt{2}$	0.07865
$2\sqrt{2}$	0.00234

Receiver output shit

$$r_o(t) = \begin{cases} s_{o1}(t) + n_o(t) & \text{code 1} \\ s_{o2}(t) + n_o(t) & \text{code 0} \end{cases}$$
$$n : \text{AWGN with } \sigma_o^2$$

ISI, channel model

Raised cosine (RC) pulse



$$0 \leq \alpha \leq 1$$

⚠ NOTE might not be safe to assume $T' = T$, if you can solve the question without T then use that method.

Nyquist criterion for zero ISI

$$D > 2W \quad \text{Use } W \text{ from table below depending on modulation scheme.}$$
$$B_{\text{Nyquist}} = \frac{W}{1+\alpha}$$
$$\alpha = \frac{\text{Excess BW}}{B_{\text{Nyquist}}} = \frac{B_{\text{abs}} - B_{\text{Nyquist}}}{B_{\text{Nyquist}}}$$

Nomenclature

$$D \rightarrow \text{Symbol Rate, Max. Signalling Rate}$$
$$T \rightarrow \text{Symbol Duration}$$
$$M \rightarrow \text{Symbol set size}$$
$$W \rightarrow \text{Bandwidth}$$

Bandwidth W and bit error rate of modulation schemes

To solve this type of question:

1. Use the formula for D below
2. Consult the BER table below to get the BER which relates the noise of the channel N_0 to E_b and to R_b.

Linear modulation	**Half**
BPSK, QPSK, M-PSK, M-QAM, ASK, FSK	M-PAM, PAM
RZ unipolar, Manchester	NRZ Unipolar, NRZ Polar, Bipolar RZ
$W = B_{\text{abs-abs}}$	$W = B_{\text{abs}}$
$W = B_{\text{abs-abs}} = \frac{1+\alpha}{T} = (1+\alpha)D$	$W = B_{\text{abs}} = \frac{1+\alpha}{2T} = (1+\alpha)D/2$
$D = \frac{W \text{ symbol/s}}{1+\alpha}$	$D = \frac{2W \text{ symbol/s}}{1+\alpha}$

$$R_b \text{ bit/s} = (D \text{ symbol/s}) \times (k \text{ bit/symbol})$$
$$M \text{ symbol/set} = 2^k$$
$$T \text{ s/symbol} = 1/(D \text{ symbol/s})$$
$$E_b = PT = P_{\text{av}}/R_b \quad \text{Energy per bit}$$

Table of bandpass signalling and BER

Binary Bandpass Signaling	$B_{\text{null-null}}$ **(Hz)**	$B_{\text{abs-abs}} = 2B_{\text{abs}}$ **(Hz)**	**BER with Coherent Detection**	**BER with Noncoherent Detection**
ASK, unipolar NRZ	$2R_b$	$R_b(1+\alpha)$	$Q\left(\sqrt{E_b/N_0}\right)$	$0.5\exp(-E_b/(2N_0))$
BPSK	$2R_b$	$R_b(1+\alpha)$	$Q\left(\sqrt{2E_b/N_0}\right)$	Requires coherent detection
Sunde's FSK	$3R_b$		$Q\left(\sqrt{E_b/N_0}\right)$	$0.5\exp(-E_b/(2N_0))$
DBPSK, M-ary Bandpass Signaling	$2R_b$	$R_b(1+\alpha)$		$0.5\exp(-E_b/N_0)$
QPSK/OQPSK ($M=4$, **PSK**)	R_b	$\frac{R_b(1+\alpha)}{2}$	$Q\left(\sqrt{2E_b/N_0}\right)$	Requires coherent detection
MSK	$1.5R_b$	$\frac{3R_b(1+\alpha)}{4}$	$Q\left(\sqrt{2E_b/N_0}\right)$	Requires coherent detection
M-PSK ($M>4$)	$2R_b/\log_2 M$	$\frac{R_b(1+\alpha)}{\log_2 M}$	$\frac{2}{\log_2 M}Q\left(\sqrt{2\log_2 M\sin^2(\pi/M)\,E_b/N_0}\right)$	Requires coherent detection
M-DPSK ($M>4$)	$2R_b/\log_2 M$	$\frac{R_b(1+\alpha)}{2\log_2 M}$		$\frac{2}{\log_2 M}Q\left(\sqrt{4\log_2 M\sin^2(\pi/(2M))\,E_b/N_0}\right)$
M-QAM (Square constellation)	$2R_b/\log_2 M$	$\frac{R_b(1+\alpha)}{\log_2 M}$	$\frac{4}{\log_2 M}\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3\log_2 M}{M-1}E_b/N_0}\right)$	Requires coherent detection
M-FSK Coherent	$\frac{(M+3)R_b}{2\log_2 M}$		$\frac{M-1}{\log_2 M}Q\left(\sqrt{(\log_2 M)E_b/N_0}\right)$	
Noncoherent	$2MR_b/\log_2 M$			$\frac{M-1}{2\log_2 M}0.5\exp(-(\log_2 M)E_b/2N_0)$

Adapted from table 11.4 M F Mesiya - Contemporary Communication Systems

PSD of modulated signals

Modulation	$G_x(f)$
Quadrature	$\frac{A_c{}^2}{4}[G_I(f-f_c)+G_I(f+f_c)+G_Q(f-f_c)+G_Q(f+f_c)]$
Linear	$\frac{\lvert V(f)\rvert^2}{2}\sum_{l=-\infty}^{\infty} R(l)\exp(-j2\pi lfT) \quad \text{What??}$

Symbol error probability

- Minimum distance between any two point
- Different from bit error since a symbol can contain multiple bits

Information theory

Stats

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(A,B)}{P(B)}$$

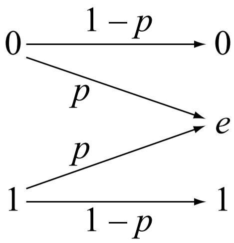
Entropy for discrete random variables

$$\begin{aligned}
H(x) &\geq 0 \\
H(x) &= -\sum_{x_i \in A_x} p_X(x_i) \log_2(p_X(x_i)) \\
H(x,y) &= -\sum_{x_i \in A_x} \sum_{y_i \in A_y} p_{XY}(x_i, y_i) \log_2(p_{XY}(x_i, y_i)) \quad \text{Joint entropy} \\
H(x,y) &= H(x) + H(y) \quad \text{Joint entropy if } x \text{ and } y \text{ independent} \\
H(x|y = y_j) &= -\sum_{x_i \in A_x} p_X(x_i|y = y_j) \log_2(p_X(x_i|y = y_j)) \quad \text{Conditional entropy} \\
H(x|y) &= -\sum_{y_j \in A_y} p_Y(y_j) H(x|y = y_j) \quad \text{Average conditional entropy, equivocation} \\
H(x|y) &= -\sum_{x_i \in A_x} \sum_{y_i \in A_y} p_X(x_i, y_j) \log_2(p_X(x_i|y = y_j)) \\
H(x|y) &= H(x,y) - H(y) \\
H(x,y) &= H(x) + H(y|x) = H(y) + H(x|y)
\end{aligned}$$

Entropy is **maximized** when all have an equal probability.

Transition probability diagram

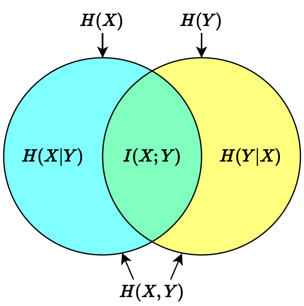
Example for binary erasure channel where X is input and Y is output:



Equivalent to:

$$\begin{aligned}
&P[Y = 0|X = 0] = 1 - p \\
&P[Y = e|X = 0] = p \\
&P[Y = 1|X = 1] = 1 - p \\
&P[Y = e|X = 1] = p \\
&P[X = 0|Y = 0] = 0 \quad \text{Note the direction} \\
&\qquad P[Y = 0] = P[Y = 0|X = 0]P[X = 0]
\end{aligned}$$

Mutual information



Amount of entropy decrease of x after observation by y.

$$I(x;y) = H(x) - H(x|y) = H(y) - H(y|x)$$

Channel model

Vertical, x: input
Horizontal, y: output
Remember $\mathbf{P}$ is a matrix where each element is $P(y_j|x_i)$

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{M1} & p_{M2} & \cdots & p_{MN} \end{bmatrix}$$

$P(y_j \mid x_i)$	y_1	y_2	$\cdots$	y_N
x_1	p_{11}	p_{12}	$\cdots$	p_{1N}
x_2	p_{21}	p_{22}	$\cdots$	p_{2N}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_M	p_{M1}	p_{M2}	$\cdots$	p_{MN}

Input has probability distribution $p_X(a_i) = P(X = a_i)$

Channel maps alphabet '$\{a_1, \dots, a_M\} \to \{b_1, \dots, b_N\}$'

Output has probability distribution $p_Y(b_j) = P(y = b_j)$

$$\begin{aligned} p_Y(b_j) &= \sum_{i=1}^{M} P[x = a_i, y = b_j] \quad 1 \le j \le N \\ &= \sum_{i=1}^{M} P[X = a_i] P[Y = b_j | X = a_i] \end{aligned}$$
$$[p_Y(b_0) \quad p_Y(b_1) \quad \dots \quad p_Y(b_j)] = [p_X(a_0) \quad p_X(a_1) \quad \dots \quad p_X(a_i)] \times \mathbf{P}$$

Fast procedure to calculate $I(y;x)$

1. Find $H(x)$
2. Find $[p_Y(b_0) \quad p_Y(b_1) \quad \dots \quad p_Y(b_j)] = [p_X(a_0) \quad p_X(a_1) \quad \dots \quad p_X(a_i)] \times \mathbf{P}$
3. Multiply each row in $\mathbf{P}$ by $p_X(a_i)$ since $p_{XY}(a_i, b_i) = P(b_i|a_i)P(a_i)$
4. Find $H(x,y)$ using each element from (3.)
5. Find $H(x|y) = H(x,y) - H(y)$
6. Find $I(x;y) = H(x) - H(x|y)$

Example of step **3**:

$$\mathbf{P_{XY}} = \begin{bmatrix} P(y_1|x_1)P(x_1) & P(y_2|x_1)P(x_1) & \cdots \\ P(y_1|x_2)P(x_2) & P(y_2|x_2)P(x_2) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Channel types

Type	Definition
Symmetric channel	Every row is a permutation of every other row, Every column is a permutation of every other column. $\text{Symmetric} \implies \text{Weakly symmetric}$
Weakly symmetric	Every row is a permutation of every other row, Every column has the same sum

Channel capacity of weakly symmetric channel

$$\begin{aligned} C &\to \text{Channel capacity (bits/channels used)} \\ N &\to \text{Output alphabet size} \\ \mathbf{p} &\to \text{Probability vector, any row of the transition matrix} \\ C &= \log_2(N) - H(\mathbf{p}) \quad \text{Capacity for weakly symmetric and symmetric channels} \\ R_b &< C \text{ for error-free transmission} \end{aligned}$$

Note that the channel capacity is realized when the channel inputs are uniformly distributed (i.e. $P(x_1) = P(x_2) = \cdots = P(x_N) = \frac{1}{N}$)

Channel capacity of an AWGN channel

$$y_i = x_i + n_i \quad n_i \sim N(0, N_0/2)$$
$$C = \frac{1}{2}\log_2\left(1 + \frac{P_{\text{av}}}{N_0/2}\right)$$

Channel capacity of a bandwidth limited AWGN channel

$$\begin{aligned} P_s &\rightarrow \text{Bandwidth limited average power} \\ y_i &= \text{bandpass}_W(x_i) + n_i \quad n_i \sim N(0, N_0/2) \\ C &= W\log_2\left(1 + \frac{P_s}{N_0 W}\right) \\ C &= W\log_2(1 + \text{SNR}) \\ \text{SNR} &= P_s/(N_0 W) \end{aligned}$$

Shannon limit

$$\begin{aligned} & R_b < C \\ \implies & R_b < W\log_2\left(1 + \frac{P_s}{N_0 W}\right) \quad \text{For bandwidth limited AWGN channel} \\ & \frac{E_b}{N_0} > \frac{2^\eta - 1}{\eta} \quad \text{SNR per bit required for error-free transmission} \\ & \eta = \frac{R_b}{W} \quad \text{Spectral efficiency (bit/(s-Hz))} \\ & \eta \gg 1 \quad \text{Bandwidth limited} \\ & \eta \ll 1 \quad \text{Power limited} \end{aligned}$$

Channel code

Note: Define XOR ($\oplus$) as exclusive OR, or modulo-2 addition.

Hamming weight	$w_H(x)$	Number of `'1'` in codeword x
Hamming distance	$d_H(x_1, x_2) = w_H(x_1 \oplus x_2)$	Number of different bits between codewords x_1 and x_2 which is the hamming weight of the XOR of the two codes.
Minimum distance	$d_{\min}$	**IMPORTANT**: $x \neq \mathbf{0}$, excludes weight of all-zero codeword. For a linear block code, $d_{\min} = w_{\min}$

Linear block code

Code is (n, k)

n is the width of a codeword

2^k codewords

A linear block code must be a subspace and satisfy both:

1. Zero vector must be present at least once
2. The XOR of any codeword pair in the code must result in a codeword that is already present in the code table.
3. $d_{\min} = w_{\min}$ (Implied by (1) and (2).)

Code generation

Each generator vector is a binary string of size n. There are k generator vectors in $\mathbf{G}$.

$$\begin{aligned}
\mathbf{g}_i &= [g_{i,0} \quad \cdots \quad g_{i,n-2} \quad g_{i,n-1}] \\
\mathbf{g}_0 &= [1010] \quad \text{Example for } n = 4 \\
\mathbf{G} &= \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_{k-1} \end{bmatrix} = \begin{bmatrix} g_{0,0} & \cdots & g_{0,n-2} & g_{0,n-1} \\ g_{1,0} & \cdots & g_{1,n-2} & g_{1,n-1} \\ \vdots & \ddots & \vdots & \vdots \\ g_{k-1,0} & \cdots & g_{k-1,n-2} & g_{k-1,n-1} \end{bmatrix}
\end{aligned}$$

A message block $\mathbf{m}$ is coded as $\mathbf{x}$ using the generation codewords in $\mathbf{G}$:

$$\begin{aligned}
\mathbf{m} &= [m_0 \quad \dots \quad m_{n-2} \quad m_{k-1}] \\
\mathbf{m} &= [101001] \quad \text{Example for } k = 6 \\
\mathbf{x} &= \mathbf{mG} = m_0\mathbf{g}_0 + m_1\mathbf{g}_1 + \cdots + m_{k-1}\mathbf{g}_{k-1}
\end{aligned}$$

Systemic linear block code

Contains k message bits (Copy $\mathbf{m}$ as-is) and $(n-k)$ parity bits after the message bits.

$$\begin{aligned}
\mathbf{G} = \; [\mathbf{I}_k \mid \mathbf{P}] \; &= \left[\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & p_{0,0} & \cdots & p_{0,n-2} & p_{0,n-1} \\ 0 & 1 & \dots & 0 & p_{1,0} & \cdots & p_{1,n-2} & p_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & p_{k-1,0} & \cdots & p_{k-1,n-2} & p_{k-1,n-1} \end{array}\right] \\
\mathbf{m} &= [m_0 \quad \dots \quad m_{n-2} \quad m_{k-1}] \\
\mathbf{x} &= \mathbf{mG} = \mathbf{m} \; [\mathbf{I}_k \mid \mathbf{P}] \; = \; [\mathbf{mI}_k \mid \mathbf{mP}] \; = \; [\mathbf{m} \mid \mathbf{b}] \\
\mathbf{b} &= \mathbf{mP} \quad \text{Parity bits of } \mathbf{x}
\end{aligned}$$

Parity check matrix $\mathbf{H}$

Transpose $\mathbf{P}$ for the parity check matrix

$$\begin{aligned}
\mathbf{H} &= \; [\mathbf{P}^{\mathrm{T}} \mid \mathbf{I}_{n-k}] \\
&= \left[\; \mathbf{p}_0{}^{\mathrm{T}} \quad \mathbf{p}_1{}^{\mathrm{T}} \quad \cdots \quad \mathbf{p}_{k-1}{}^{\mathrm{T}} \mid \mathbf{I}_{n-k} \; \right] \\
&= \left[\begin{array}{cccc|cccc} p_{0,0} & \cdots & p_{0,k-2} & p_{0,k-1} & 1 & 0 & \dots & 0 \\ p_{1,0} & \cdots & p_{1,k-2} & p_{1,k-1} & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1,0} & \cdots & p_{n-1,k-2} & p_{n-1,k-1} & 0 & 0 & \dots & 1 \end{array}\right] \\
\mathbf{xH}^{\mathrm{T}} = \mathbf{0} &\implies \text{Codeword is valid}
\end{aligned}$$

Procedure to find parity check matrix from list of codewords

1. From the number of codewords, find $k = \log_2(N)$
2. Partition codewords into k information bits and remaining bits into $n-k$ parity bits. The information bits should be a simple counter (?).
3. Express parity bits as a linear combination of information bits
4. Put coefficients into $\mathbf{P}$ matrix and find $\mathbf{H}$

Example:

x_1	x_2	x_3	x_4	x_5
1	0	1	1	0
0	1	1	1	1
0	0	0	0	0
1	1	0	0	1

Set x_1, x_2 as information bits. Express x_3, x_4, x_5 in terms of x_1, x_2.

$$\begin{aligned} x_3 &= x_1 \oplus x_2 \\ x_4 &= x_1 \oplus x_2 \\ x_5 &= x_2 \end{aligned} \implies \mathbf{P} = \begin{array}{c|cc} & x_1 & x_2 \\ \hline x_3 & 1 & 1 \\ x_4 & 1 & 1 \\ x_5 & 0 & 1 \end{array}$$

$$\mathbf{H} = \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Error detection and correction

Detection of s errors: $d_{\min} \geq s+1$

Correction of u errors: $d_{\min} \geq 2u+1$

CHECKLIST

- Transfer function in complex envelope form $\tilde{h}(t)$ should be divided by two.
- Convolutions: do not forget width when using graphical method
- $2W$ for rectangle functions
- Scale sampled spectrum by f_s
- $2f_c$ for spectrum after IF mixing.
- Square transfer function for PSD $G_y(f) = |H(f)|^2 G_x(f)$
- Square besselJ function for FM power $|J_n(\beta)|^2$
- Bandwidth: only consider positive frequencies (so the bandwidth of an AM signal will be the range from the lowest to greatest sideband frequency. For a rectangular function, it will be from 0 to W).
- TODO: add more items to check
- TODO: add some graphics for these checklist items