Idiot's guide to ELEC4402 communication systems

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Fourier transform identities

Time Function	Fourier Transform
$\text{rect}\left(\frac{t}{T}\right) \quad \Pi\left(\frac{t}{T}\right)$	$T\text{sinc}(fT)$
$\text{sinc}(2Wt)$	$\frac{1}{2W}\text{rect}\left(\frac{f}{2W}\right) \quad \frac{1}{2W}\Pi\left(\frac{f}{2W}\right)$
$\exp(-at)u(t), \quad a>0$	$\frac{1}{a+j2\pi f}$
$\exp(-a\|t\|), \quad a>0$	$\frac{2a}{a^2+(2\pi f)^2}$
$\exp(-\pi t^2)$	$\exp(-\pi f^2)$
$1-\frac{\|t\|}{T}, \quad \|t\|<T$	$T\text{sinc}^2(fT)$
$\delta(t)$	1
1	$\delta(f)$
$\delta(t-t_0)$	$\exp(-j2\pi f t_0)$
$g(t-a)$	$\exp(-j2\pi fa)G(f) \quad \text{shift property}$
$g(bt)$	$\frac{G(f/b)}{\|b\|} \quad \text{scaling property}$
$g(bt-a)$	$\frac{1}{\|b\|}\exp(-j2\pi a(f/b))\cdot G(f/b) \quad \text{shift and scale}$
$\frac{d}{dt}g(t)$	$j2\pi fG(f) \quad \text{differentiation property}$
$G(t)$	$g(-f) \quad \text{duality property}$
$g(t)h(t)$	$G(f)*H(f)$
$g(t)*h(t)$	$G(f)H(f)$
$\exp(j2\pi f_c t)$	$\delta(f-f_c)$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f-f_c)+\delta(f+f_c)]$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f-f_c)-\delta(f+f_c)]$
$\text{sgn}(t)$	$\frac{1}{j\pi f}$
$\frac{1}{\pi t}$	$-j\text{sgn}(f)$
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)=f_0\sum_{n=-\infty}^{\infty}\delta\left(f-nf_0\right)$

$$u(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases} \qquad \text{Unit Step Function}$$

$$\text{sgn}(t) = \begin{cases} +1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases} \qquad \text{Signum Function}$$

$$\text{sinc}(2Wt) = \frac{\sin(2\pi Wt)}{2\pi Wt} \qquad \text{sinc Function}$$

$$\text{rect}(t) = \Pi(t) = \begin{cases} 1, & -0.5 < t < 0.5 \\ 0, & |t| > 0.5 \end{cases} \qquad \text{Rectangular/Gate Function}$$

$$g(t) * h(t) = (g * h)(t) = \int_{\infty}^{\infty} g(\tau)h(t-\tau)d\tau \qquad \text{Convolution}$$

Fourier transform of continuous time periodic signal

Required for some questions on **sampling**:

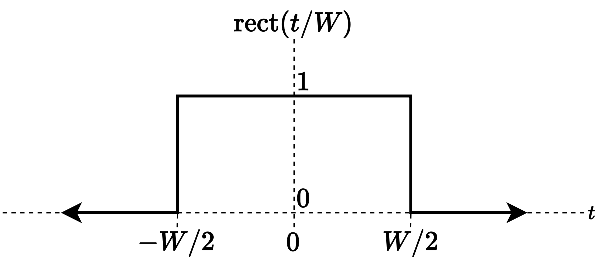
Transform a continuous time-periodic signal $x_p(t) = \sum_{n=-\infty}^{\infty} x(t - nT_s)$ with period T_s:

$$X_p(f) = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_s) \quad f_s = \frac{1}{T_s}$$

Calculate C_n coefficient as follows from $x_p(t)$:

$$\begin{aligned} C_n &= \frac{1}{T_s} \int_{T_s} x_p(t) \exp(-j2\pi f_s t) dt \\ &= \frac{1}{T_s} X(nf_s) \quad \text{(TODO: Check)} \quad x(t - nT_s) \text{ is contained in the interval } T_s \end{aligned}$$

rect function



Bessel function

$$\sum_{n \in \mathbb{Z}} {J_n}^2(\beta) = 1$$

$$J_n(\beta) = (-1)^n J_{-n}(\beta)$$

White noise

$$\begin{aligned} R_W(\tau) &= \frac{N_0}{2}\delta(\tau) = \frac{kT}{2}\delta(\tau) = \sigma^2\delta(\tau) \\ G_w(f) &= \frac{N_0}{2} \\ N_0 &= kT \\ G_y(f) &= |H(f)|^2 G_w(f) \\ G_y(f) &= G(f)G_w(f) \end{aligned}$$

WSS

$$\begin{aligned} \mu_X(t) &= \mu_X \text{ Constant} \\ R_{XX}(t_1, t_2) &= R_X(t_1 - t_2) = R_X(\tau) \\ E[X(t_1)X(t_2)] &= E[X(t)X(t+\tau)] \end{aligned}$$

Ergodicity

$$\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} x(t)dt$$
$$\langle X(t+\tau)X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} x(t+\tau)x(t)dt$$
$$E[\langle X(t)\rangle_T] = \frac{1}{2T}\int_{-T}^{T} x(t)dt = \frac{1}{2T}\int_{-T}^{T} m_X dt = m_X$$

Type	Normal	Mean square sense
ergodic in mean	$\lim_{T\to\infty} \langle X(t)\rangle_T = m_X(t) = m_X$	$\lim_{T\to\infty} \mathrm{VAR}[\langle X(t)\rangle_T] = 0$
ergodic in autocorrelation function	$\lim_{T\to\infty} \langle X(t+\tau)X(t)\rangle_T = R_X(\tau)$	$\lim_{T\to\infty} \mathrm{VAR}[\langle X(t+\tau)X(t)\rangle_T] = 0$

A WSS random process needs to be both ergodic in mean and autocorrelation to be considered an ergodic process

Other identities

$$f*(g*h) = (f*g)*h \quad \text{Convolution associative}$$
$$a(f*g) = (af)*g \quad \text{Convolution associative}$$
$$\sum_{x=-\infty}^{\infty} (f(xa)\delta(\omega - xb)) = f\left(\frac{\omega a}{b}\right)$$

Other trig

$$\cos 2\theta = 2\cos^2\theta - 1 \Leftrightarrow \frac{\cos 2\theta + 1}{2} = \cos^2\theta$$
$$e^{-j\alpha} - e^{j\alpha} = -2j\sin(\alpha)$$
$$e^{-j\alpha} + e^{j\alpha} = 2\cos(\alpha)$$
$$\cos(-A) = \cos(A)$$
$$\sin(-A) = -\sin(A)$$
$$\sin(A+\pi/2) = \cos(A)$$
$$\sin(A-\pi/2) = -\cos(A)$$
$$\cos(A-\pi/2) = \sin(A)$$
$$\cos(A+\pi/2) = -\sin(A)$$
$$\int_{x\in\mathbb{R}} \mathrm{sinc}(Ax) = \frac{1}{|A|}$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\cos(A)\cos(B) = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$
$$\cos(A)\sin(B) = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$
$$\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A}{2} - \frac{B}{2}\right)\cos\left(\frac{A}{2} + \frac{B}{2}\right)$$
$$\cos(A) - \cos(B) = -2\sin\left(\frac{A}{2} - \frac{B}{2}\right)\sin\left(\frac{A}{2} + \frac{B}{2}\right)$$
$$\sin(A) + \sin(B) = 2\sin\left(\frac{A}{2} + \frac{B}{2}\right)\cos\left(\frac{A}{2} - \frac{B}{2}\right)$$
$$\sin(A) - \sin(B) = 2\sin\left(\frac{A}{2} - \frac{B}{2}\right)\cos\left(\frac{A}{2} + \frac{B}{2}\right)$$
$$\cos(A) + \sin(B) = -2\sin\left(\frac{A}{2} - \frac{B}{2} - \frac{\pi}{4}\right)\sin\left(\frac{A}{2} + \frac{B}{2} + \frac{\pi}{4}\right)$$
$$\cos(A) - \sin(B) = -2\sin\left(\frac{A}{2} + \frac{B}{2} - \frac{\pi}{4}\right)\sin\left(\frac{A}{2} - \frac{B}{2} + \frac{\pi}{4}\right)$$

IQ/Complex envelope

Def. $\tilde{g}(t) = g_I(t) + jg_Q(t)$ as the complex envelope. Best to convert to $e^{j\theta}$ form.

Convert complex envelope representation to time-domain representation of signal

$$\begin{aligned} g(t) &= g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t) \\ &= \mathrm{Re}[\tilde{g}(t)\exp(j2\pi f_c t)] \\ &= A(t)\cos(2\pi f_c t + \phi(t)) \\ A(t) &= |g(t)| = \sqrt{g_I^2(t) + g_Q^2(t)} \quad \text{Amplitude} \\ \phi(t) & \quad \text{Phase} \\ g_I(t) &= A(t)\cos(\phi(t)) \quad \text{In-phase component} \\ g_Q(t) &= A(t)\sin(\phi(t)) \quad \text{Quadrature-phase component} \end{aligned}$$

For transfer function

$$\begin{aligned} h(t) &= h_I(t)\cos(2\pi f_c t) - h_Q(t)\sin(2\pi f_c t) \\ &= 2\mathrm{Re}[\tilde{h}(t)\exp(j2\pi f_c t)] \\ \Rightarrow \tilde{h}(t) &= h_I(t)/2 + jh_Q(t)/2 = A(t)/2\exp(j\phi(t)) \end{aligned}$$

AM

CAM

$$\begin{aligned} m_a &= \frac{\min_t |k_a m(t)|}{A_c} \quad k_a \text{ is the amplitude sensitivity (volt}^{-1}\text{)}, m_a \text{ is the modulation index.} \\ m_a &= \frac{A_{\max} - A_{\min}}{A_{\max} + A_{\min}} \quad (\text{Symmetrical } m(t)) \\ m_a &= k_a A_m \quad (\text{Symmetrical } m(t)) \\ x(t) &= A_c\cos(2\pi f_c t)\left[1 + k_a m(t)\right] = A_c\cos(2\pi f_c t)\left[1 + m_a m(t)/A_c\right], \\ & \text{where } m(t) = A_m\hat{m}(t) \text{ and } \hat{m}(t) \text{ is the normalized modulating signal} \\ P_c &= \frac{{A_c}^2}{2} \quad \text{Carrier power} \\ P_x &= \frac{1}{4}{m_a}^2{A_c}^2 \\ \eta &= \frac{\text{Signal Power}}{\text{Total Power}} = \frac{P_x}{P_x + P_c} \\ B_T &= 2f_m = 2B \end{aligned}$$

B_T: Signal bandwidth B: Bandwidth of modulating wave

Overmodulation (resulting in phase reversals at crossing points): $m_a > 1$

DSB-SC

$$\begin{aligned} x_{\mathrm{DSB}}(t) &= A_c\cos(2\pi f_c t)m(t) \\ B_T &= 2f_m = 2B \end{aligned}$$

FM/PM

$$\begin{aligned} s(t) &= A_c\cos\left[2\pi f_c t + k_p m(t)\right] \quad \text{Phase modulated (PM)} \\ s(t) &= A_c\cos\left[2\pi f_c t + 2\pi k_f\int_0^t m(\tau)d\tau\right] \quad \text{Frequency modulated (FM)} \\ s(t) &= A_c\cos\left[2\pi f_c t + \beta\sin(2\pi f_m t)\right] \quad \text{FM single tone} \\ \beta &= \frac{\Delta f}{f_m} = k_f A_m \quad \text{Modulation index} \\ \Delta f &= \beta f_m = k_f A_m f_m = \max_t(k_f m(t)) - \min_t(k_f m(t)) \quad \text{Maximum frequency deviation} \\ D &= \frac{\Delta f}{W_m} \quad \text{Deviation ratio, where } W_m \text{ is bandwidth of } m(t) \text{ (Use FT)} \end{aligned}$$

Bessel form and magnitude spectrum (single tone)

$$s(t) = A_c\cos\left[2\pi f_c t + \beta\sin(2\pi f_m t)\right] \Leftrightarrow s(t) = A_c\sum_{n=-\infty}^{\infty} J_n(\beta)\cos[2\pi(f_c + nf_m)t]$$

FM signal power

$$
\begin{aligned}
P_{\text{av}} &= \frac{{A_c}^2}{2} \\
P_{\text{band\_index}} &= \frac{{A_c}^2 {J_{\text{band\_index}}}^2(\beta)}{2} \\
\text{band\_index} = 0 &\implies f_c + 0 f_m \\
\text{band\_index} = 1 &\implies f_c + 1 f_m, \ldots
\end{aligned}
$$

Carson's rule to find B (98% power bandwidth rule)

$$
\begin{aligned}
B &= 2 M f_m = 2(\beta + 1) f_m \\
&= 2(\Delta f + f_m) \\
&= 2(k_f A_m + f_m) \\
&= 2(D + 1) W_m \\
B &= \begin{cases} 2(\Delta f + f_m) & \text{FM, sinusoidal message} \\ 2(\Delta \phi + 1) f_m & \text{PM, sinusoidal message} \end{cases}
\end{aligned}
$$

Δf of arbitrary modulating signal

Find instantaneous frequency f_{FM}.

M: Number of **pairs** of significant sidebands

$$
\begin{aligned}
s(t) &= A_c \cos(\theta_{\text{FM}}(t)) \\
f_{\text{FM}}(t) &= \frac{1}{2\pi} \frac{d\theta_{\text{FM}}(t)}{dt} \\
A_m &= \max_t |m(t)| \\
\Delta f &= \max_t (f_{\text{FM}}(t)) - f_c \\
W_m &= \max(\text{frequencies in } \theta_{\text{FM}}(t) \ldots) \\
\text{Example: } \operatorname{sinc}(At + t) + 2\cos(2\pi t) &= \frac{\sin(2\pi((At + t)/2))}{\pi(At + t)} + 2\cos(2\pi t) \to W_m = \max\left(\frac{A+1}{2}, 1\right) \\
D &= \frac{\Delta f}{W_m} \\
B_T &= 2(D + 1) W_m
\end{aligned}
$$

Complex envelope

$$
\begin{aligned}
s(t) &= A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \Leftrightarrow \tilde{s}(t) = A_c \exp(j\beta \sin(2\pi f_m t)) \\
s(t) &= \operatorname{Re}[\tilde{s}(t) \exp(j 2\pi f_c t)] \\
\tilde{s}(t) &= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \exp(j 2\pi f_m t)
\end{aligned}
$$

Band

Narrowband	Wideband
$D < 1, \beta < 1$	$D > 1, \beta > 1$

Power, energy and autocorrelation

$$
\begin{aligned}
G_{\mathrm{WGN}}(f) &= \frac{N_0}{2} \\
G_x(f) &= |H(f)|^2 G_w(f) \text{ (PSD)} \\
G_x(f) &= G(f) G_w(f) \text{ (PSD)} \\
G_x(f) &= \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} \text{ (PSD)} \\
G_x(f) &= \mathfrak{F}[R_x(\tau)] \text{ (WSS)} \\
P = \sigma^2 &= \int_{\mathbb{R}} G_x(f) df \\
P = \sigma^2 &= \lim_{t\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt \\
P[A\cos(2\pi f t + \phi)] &= \frac{A^2}{2} \quad \text{Power of sinusoid} \\
E &= \int_{-\infty}^{\infty} |x(t)|^2 dt = |X(f)|^2 \\
R_x(\tau) &= \mathfrak{F}(G_x(f)) \quad \text{PSD to Autocorrelation}
\end{aligned}
$$

Noise performance

$$
\begin{aligned}
\mathrm{CNR}_{\mathrm{in}} &= \frac{P_{\mathrm{in}}}{P_{\mathrm{noise}}} \\
\mathrm{CNR}_{\mathrm{in,FM}} &= \frac{A^2}{2WN_0} \\
\mathrm{SNR}_{\mathrm{FM}} &= \frac{3A^2 k_f^2 P}{2N_0 W^3} \\
\mathrm{SNR(dB)} &= 10\log_{10}(\mathrm{SNR}) \quad \text{Decibels from ratio}
\end{aligned}
$$

Sampling

$$
\begin{aligned}
t &= nT_s \\
T_s &= \frac{1}{f_s} \\
x_s(t) &= x(t)\delta_s(t) = x(t)\sum_{n\in\mathbb{Z}} \delta(t - nT_s) = \sum_{n\in\mathbb{Z}} x(nT_s)\delta(t - nT_s) \\
X_s(f) &= X(f) * \sum_{n\in\mathbb{Z}} \delta\left(f - \frac{n}{T_s}\right) = X(f) * \sum_{n\in\mathbb{Z}} \delta(f - nf_s) \\
B &> \frac{1}{2} f_s, 2B > f_s \to \text{Aliasing}
\end{aligned}
$$

Procedure to reconstruct sampled signal

Analog signal $x'(t)$ which can be reconstructed from a sampled signal $x_s(t)$: Put $x_s(t)$ through LPF with maximum frequency of $f_s/2$ and minimum frequency of $-f_s/2$. Anything outside of the BPF will be attenuated, therefore n which results in frequencies outside the BPF will evaluate to 0 and can be ignored.

Example: $f_s = 5000 \implies \text{LPF} \in [-2500, 2500]$

Then iterate for $n = 0, 1, -1, 2, -2, \ldots$ until the first iteration where the result is 0 since all terms are eliminated by the LPF.

TODO: Add example

Then add all terms and transform $\bar{X}_s(f)$ back to time domain to get $x_s(t)$

Fourier transform of continuous time periodic signal (1)

Required for some questions on **sampling**:

Transform a continuous time-periodic signal $x_p(t) = \sum_{n=-\infty}^{\infty} x(t - nT_s)$ with period T_s:

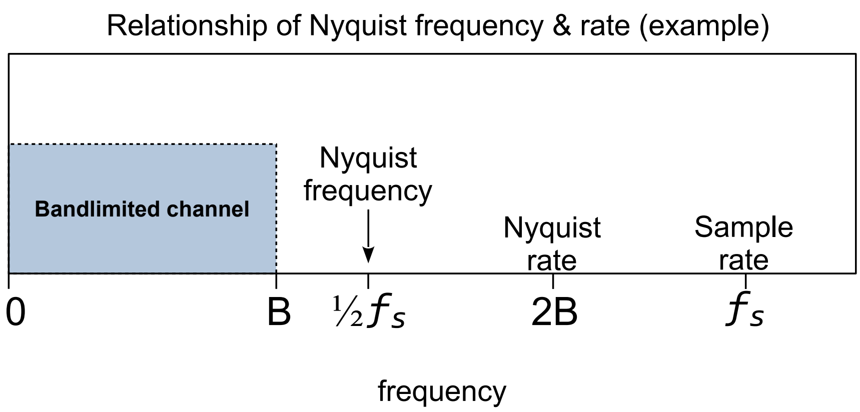
$$
X_p(f) = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_s) \quad f_s = \frac{1}{T_s}
$$

Calculate C_n coefficient as follows from $x_p(t)$:

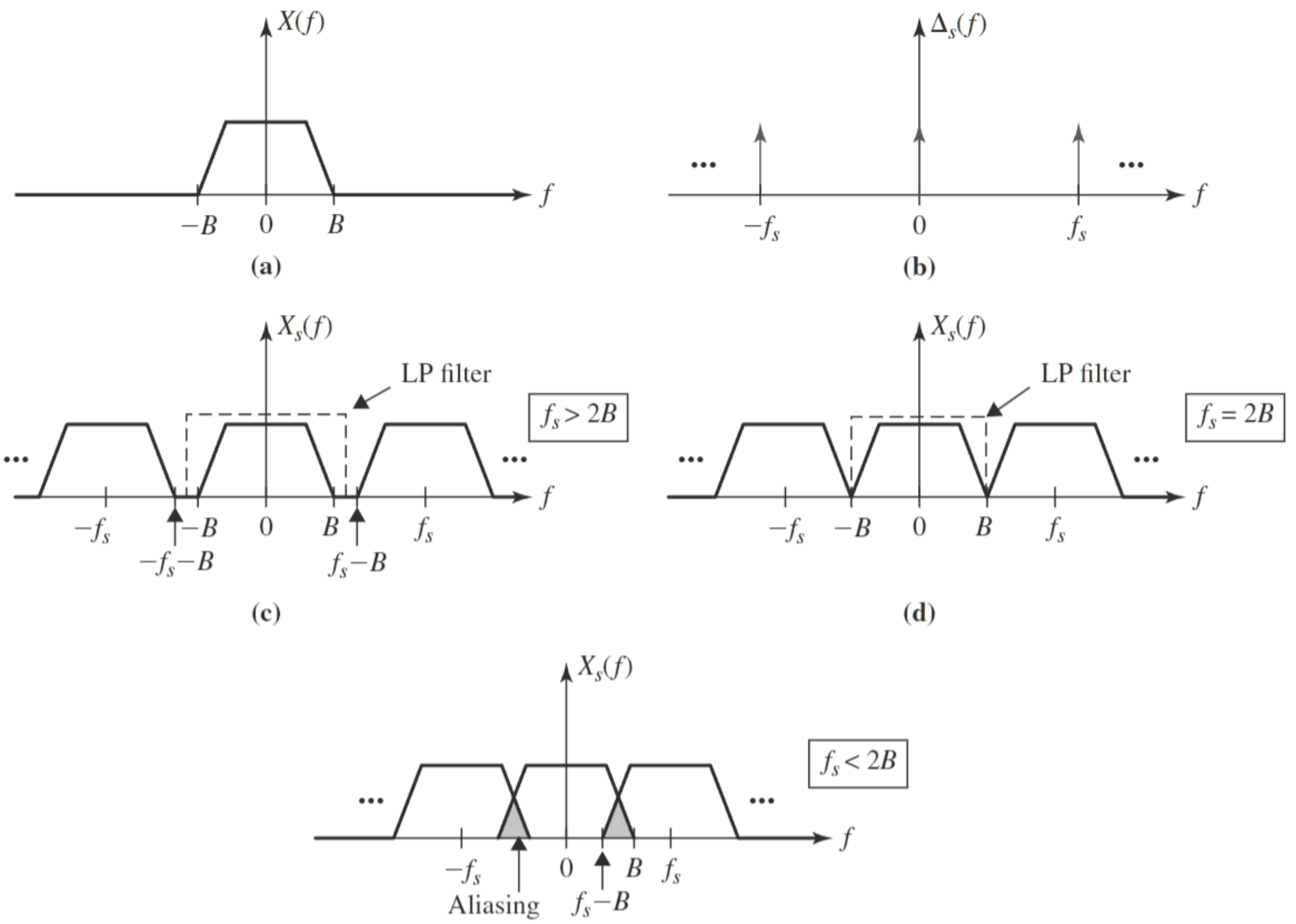
$$
\begin{aligned}
C_n &= \frac{1}{T_s}\int_{T_s} x_p(t)\exp(-j2\pi f_s t)dt \\
&= \frac{1}{T_s}X(nf_s) \quad \text{(TODO: Check)} \quad x(t-nT_s) \text{ is contained in the interval } T_s
\end{aligned}
$$

Nyquist criterion for zero-ISI

Do not transmit more than $2B$ samples per second over a channel of B bandwidth.



Insert here figure 8.3 from M F Mesiya - Contemporary Communication Systems (Add image to `images/sampling.png`)



sampling

Quantizer

$$
\Delta = \frac{x_{\text{Max}} - x_{\text{Min}}}{2^k} \quad \text{for } k\text{-bit quantizer (V/lsb)}
$$

Quantization noise

$$
\begin{aligned}
e &:= y - x \quad \text{Quantization error} \\
\mu_E &= E[E] = 0 \quad \text{Zero mean} \\
\sigma_E{}^2 &= E[E^2] - 0^2 = \int_{-\Delta/2}^{\Delta/2} e^2 \times \left(\frac{1}{\Delta}\right) de \quad \text{Where } E \sim 1/\Delta \text{ uniform over } (-\Delta/2, \Delta/2) \\
\text{SQNR} &= \frac{\text{Signal power}}{\text{Quantization noise}} \\
\text{SQNR(dB)} &= 10\log_{10}(\text{SQNR})
\end{aligned}
$$

Insert here figure 8.17 from M F Mesiya - Contemporary Communication Systems (Add image to `images/quantizer.png`)

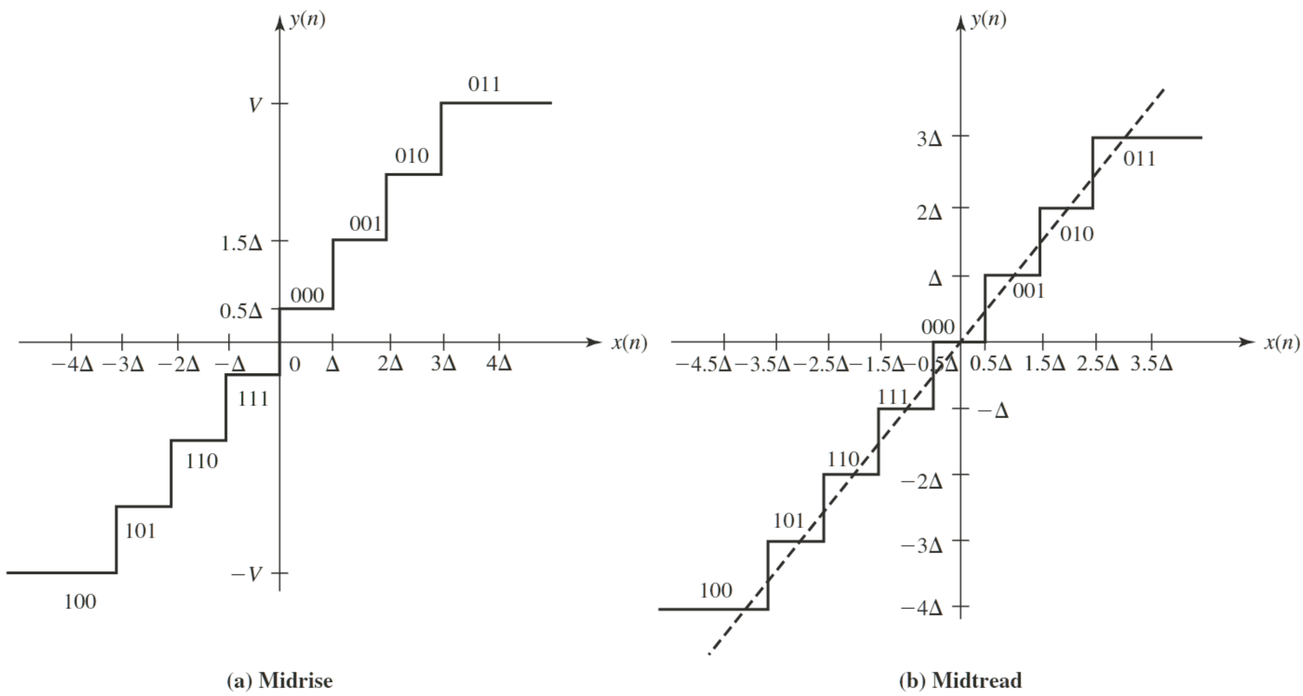
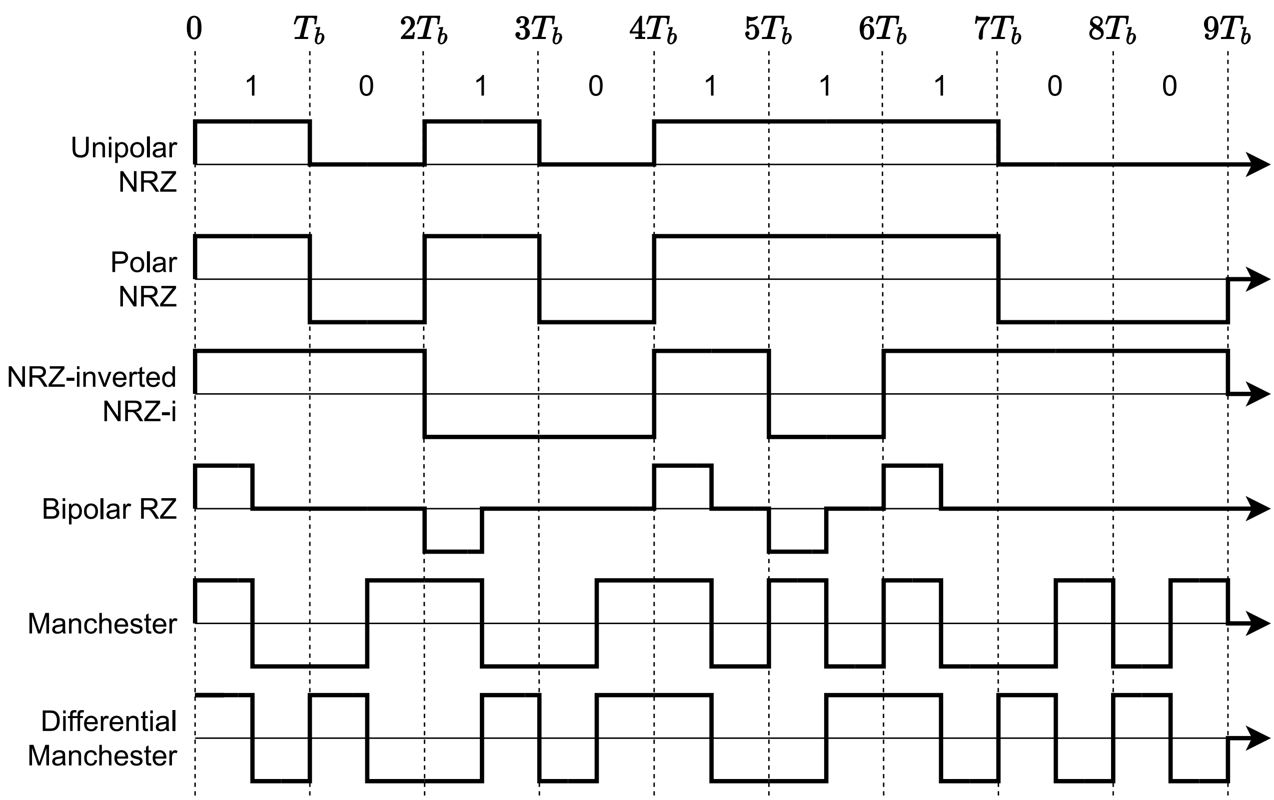


Figure 8.17 Two types of eight-level uniform quantizer.

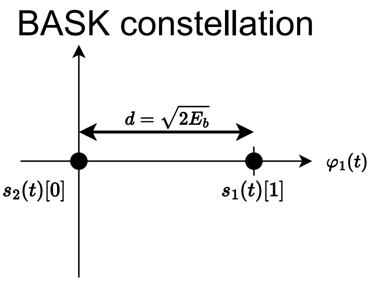
quantizer

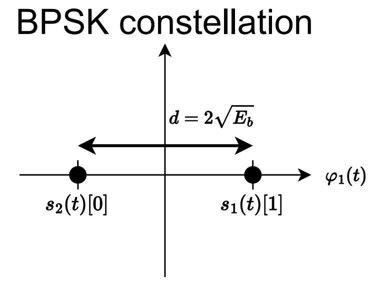
Line codes

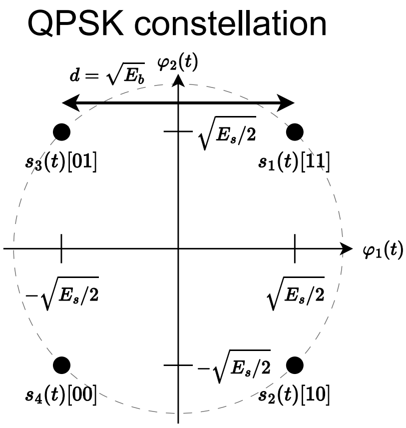


$$R_b \to \text{Bit rate}$$
$$D \to \text{Symbol rate} \mid R_d \mid 1/T_b$$
$$A \to m_a$$
$$V(f) \to \text{Pulse shape}$$
$$V_{\text{rectangle}}(f) = T\text{sinc}(fT \times \text{DutyCycle})$$
$$G_{\text{MunipolarNRZ}}(f) = \frac{(M^2-1)A^2D}{12}|V(f)|^2 + \frac{(M-1)^2}{4}(DA)^2\sum_{l=-\infty}^{\infty}|V(lD)|^2\delta(f-lD)$$
$$G_{\text{MpolarNRZ}}(f) = \frac{(M^2-1)A^2D}{3}|V(f)|^2$$
$$G_{\text{unipolarNRZ}}(f) = \frac{A^2}{4R_b}\left(\text{sinc}^2\left(\frac{f}{R_b}\right) + R_b\delta(f)\right), \text{NB}_0 = R_b$$
$$G_{\text{polarNRZ}}(f) = \frac{A^2}{R_b}\text{sinc}^2\left(\frac{f}{R_b}\right)$$
$$G_{\text{unipolarNRZ}}(f) = \frac{A^2}{4R_b}\left(\text{sinc}^2\left(\frac{f}{R_b}\right) + R_b\delta(f)\right)$$
$$G_{\text{unipolarRZ}}(f) = \frac{A^2}{16}\left(\sum_{l=-\infty}^{\infty}\delta\left(f-\frac{l}{T_b}\right)\left|\text{sinc}(\text{duty}\times l)\right|^2 + T_b\left|\text{sinc}\left(\text{duty}\times fT_b\right)\right|^2\right), \text{NB}_0 = 2R_b$$

Modulation and basis functions







BASK

Basis functions

$$\varphi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t) \quad 0 \le t \le T_b$$

Symbol mapping

$$b_n : \{1, 0\} \to a_n : \{1, 0\}$$

2 possible waveforms

$$s_1(t) = A_c\sqrt{\frac{T_b}{2}}\varphi_1(t) = \sqrt{2E_b}\varphi_1(t)$$
$$s_1(t) = 0$$
$$\text{Since } E_b = E_{\text{average}} = \tfrac{1}{2}(\tfrac{A_c{}^2}{2} \times T_b + 0) = \tfrac{A_c{}^2}{4}T_b$$

Distance is $d = \sqrt{2E_b}$

BPSK

Basis functions

$$\varphi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t) \quad 0 \le t \le T_b$$

Symbol mapping

$$b_n : \{1, 0\} \rightarrow a_n : \{1, -1\}$$

2 possible waveforms

$$\begin{aligned} s_1(t) &= A_c\sqrt{\frac{T_b}{2}}\varphi_1(t) = \sqrt{E_b}\varphi_1(t) \\ s_1(t) &= -A_c\sqrt{\frac{T_b}{2}}\varphi_1(t) = -\sqrt{E_b}\varphi_2(t) \\ &\text{Since } E_b = E_{\text{average}} = \tfrac{1}{2}(\tfrac{{A_c}^2}{2} \times T_b + \tfrac{{A_c}^2}{2} \times T_b) = \tfrac{{A_c}^2}{2}T_b \end{aligned}$$

Distance is $d = 2\sqrt{E_b}$

QPSK ($M = 4$ PSK)

Basis functions

$$\begin{aligned} T &= 2T_b \quad \text{Time per symbol for two bits } T_b \\ \varphi_1(t) &= \sqrt{\frac{2}{T}}\cos(2\pi f_c t) \quad 0 \le t \le T \\ \varphi_2(t) &= \sqrt{\frac{2}{T}}\sin(2\pi f_c t) \quad 0 \le t \le T \end{aligned}$$

4 possible waveforms

$$\begin{aligned} s_1(t) &= \sqrt{E_s/2}\,[\varphi_1(t) + \varphi_2(t)] \\ s_2(t) &= \sqrt{E_s/2}\,[\varphi_1(t) - \varphi_2(t)] \\ s_3(t) &= \sqrt{E_s/2}\,[-\varphi_1(t) + \varphi_2(t)] \\ s_4(t) &= \sqrt{E_s/2}\,[-\varphi_1(t) - \varphi_2(t)] \end{aligned}$$

Note on energy per symbol: Since $|s_i(t)| = A_c$, have to normalize distance as follows:

$$\begin{aligned} s_i(t) &= A_c\sqrt{T/2}/\sqrt{2} \times [\alpha_{1i}\varphi_1(t) + \alpha_{2i}\varphi_2(t)] \\ &= \sqrt{T{A_c}^2/4}\,[\alpha_{1i}\varphi_1(t) + \alpha_{2i}\varphi_2(t)] \\ &= \sqrt{E_s/2}\,[\alpha_{1i}\varphi_1(t) + \alpha_{2i}\varphi_2(t)] \end{aligned}$$

Signal

$$\begin{aligned} \text{Symbol mapping: } &\{1, 0\} \rightarrow \{1, -1\} \\ I(t) &= b_{2n}\varphi_1(t) \quad \text{Even bits} \\ Q(t) &= b_{2n+1}\varphi_2(t) \quad \text{Odd bits} \\ x(t) &= A_c[I(t)\cos(2\pi f_c t) - Q(t)\sin(2\pi f_c t)] \end{aligned}$$

Example of waveform

▸ Code

Remember that $T = 2T_b$

b_n	QPSK bits
$I(t)$ (Odd, 1st bits)	QPSK bits
$Q(t)$ (Even, 2nd bits)	QPSK bits

Matched filter

1. Filter function

Find transfer function $h(t)$ of matched filter and apply to an input:

$$h(t) = s_1(T-t) - s_2(T-t)$$
$$h(t) = s^*(T-t) \qquad ((.)^* \text{ is the conjugate})$$
$$s_{on}(t) = h(t) * s_n(t) = \int_{\infty}^{\infty} h(\tau) s_n(t-\tau) d\tau \quad \text{Filter output}$$
$$n_o(t) = h(t) * n(t) \quad \text{Noise at filter output}$$

2. Bit error rate

Bit error rate (BER) from matched filter outputs and filter output noise

$$Q(x) = \frac{1}{2} - \frac{1}{2}\text{erf}\left(\frac{x}{\sqrt{2}}\right) \Leftrightarrow \text{erf}\left(\frac{x}{\sqrt{2}}\right) = 1 - 2Q(x)$$
$$E_b = d^2 = \int_{-\infty}^{\infty} |s_1(t) - s_2(t)|^2 dt \quad \text{Energy per bit/Distance}$$
$$T = 1/R_b \quad R_b\text{: Bitrate}$$
$$E_b = PT = P_{\text{av}}/R_b \quad \text{Energy per bit}$$
$$P(\text{W}) = 10^{\frac{P(\text{dB})}{10}}$$
$$P_{\text{RX}}(W) = P_{\text{TX}}(W) \cdot 10^{\frac{P_{\text{loss}}(\text{dB})}{10}} \qquad P_{\text{loss}} \text{ is expressed with negative sign e.g. "-130 dB"}$$
$$\text{BER}_{\text{MatchedFilter}} = Q\left(\sqrt{\frac{d^2}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{2N_0}}\right)$$
$$\text{BER}_{\text{unipolarNRZ|BASK}} = Q\left(\sqrt{\frac{d^2}{N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$
$$\text{BER}_{\text{polarNRZ|BPSK}} = Q\left(\sqrt{\frac{2d^2}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Value tables for $\mathrm{erf}(x)$ and $Q(x)$

$\mathrm{erf}(x)$ function

x	$\mathrm{erf}(x)$	x	$\mathrm{erf}(x)$	x	$\mathrm{erf}(x)$
0.00	0.00000	0.75	0.71116	1.50	0.96611
0.05	0.05637	0.80	0.74210	1.55	0.97162
0.10	0.11246	0.85	0.77067	1.60	0.97635
0.15	0.16800	0.90	0.79691	1.65	0.98038
0.20	0.22270	0.95	0.82089	1.70	0.98379
0.25	0.27633	1.00	0.84270	1.75	0.98667
0.30	0.32863	1.05	0.86244	1.80	0.98909
0.35	0.37938	1.10	0.88021	1.85	0.99111
0.40	0.42839	1.15	0.89612	1.90	0.99279
0.45	0.47548	1.20	0.91031	1.95	0.99418
0.50	0.52050	1.25	0.92290	2.00	0.99532
0.55	0.56332	1.30	0.93401	2.50	0.99959
0.60	0.60386	1.35	0.94376	3.00	0.99998
0.65	0.64203	1.40	0.95229	3.30	0.999998**
0.70	0.67780	1.45	0.95970		

$Q(x)$ function

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0.00	0.5	2.30	0.010724	4.55	2.6823×10^{-6}	6.80	5.231×10^{-12}
0.05	0.48006	2.35	0.0093867	4.60	2.1125×10^{-6}	6.85	3.6925×10^{-12}
0.10	0.46017	2.40	0.0081975	4.65	1.6597×10^{-6}	6.90	2.6001×10^{-12}
0.15	0.44038	2.45	0.0071428	4.70	1.3008×10^{-6}	6.95	1.8264×10^{-12}
0.20	0.42074	2.50	0.0062097	4.75	1.0171×10^{-6}	7.00	1.2798×10^{-12}
0.25	0.40129	2.55	0.0053861	4.80	7.9333×10^{-7}	7.05	8.9459×10^{-13}
0.30	0.38209	2.60	0.0046612	4.85	6.1731×10^{-7}	7.10	6.2378×10^{-13}
0.35	0.36317	2.65	0.0040246	4.90	4.7918×10^{-7}	7.15	4.3389×10^{-13}
0.40	0.34458	2.70	0.003467	4.95	3.7107×10^{-7}	7.20	3.0106×10^{-13}
0.45	0.32636	2.75	0.0029798	5.00	2.8665×10^{-7}	7.25	2.0839×10^{-13}
0.50	0.30854	2.80	0.0025551	5.05	2.2091×10^{-7}	7.30	1.4388×10^{-13}
0.55	0.29116	2.85	0.002186	5.10	1.6983×10^{-7}	7.35	9.9103×10^{-14}
0.60	0.27425	2.90	0.0018658	5.15	1.3024×10^{-7}	7.40	6.8092×10^{-14}
0.65	0.25785	2.95	0.0015889	5.20	9.9644×10^{-8}	7.45	4.667×10^{-14}
0.70	0.24196	3.00	0.0013499	5.25	7.605×10^{-8}	7.50	3.1909×10^{-14}
0.75	0.22663	3.05	0.0011442	5.30	5.7901×10^{-8}	7.55	2.1763×10^{-14}
0.80	0.21186	3.10	0.0009676	5.35	4.3977×10^{-8}	7.60	1.4807×10^{-14}
0.85	0.19766	3.15	0.00081635	5.40	3.332×10^{-8}	7.65	1.0049×10^{-14}
0.90	0.18406	3.20	0.00068714	5.45	2.5185×10^{-8}	7.70	6.8033×10^{-15}
0.95	0.17106	3.25	0.00057703	5.50	1.899×10^{-8}	7.75	4.5946×10^{-15}
1.00	0.15866	3.30	0.00048342	5.55	1.4283×10^{-8}	7.80	3.0954×10^{-15}
1.05	0.14686	3.35	0.00040406	5.60	1.0718×10^{-8}	7.85	2.0802×10^{-15}

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
1.10	0.13567	3.40	0.00033693	5.65	8.0224×10^{-9}	7.90	1.3945×10^{-15}
1.15	0.12507	3.45	0.00028029	5.70	5.9904×10^{-3}	7.95	9.3256×10^{-16}
1.20	0.11507	3.50	0.00023263	5.75	4.4622×10^{-9}	8.00	6.221×10^{-16}
1.25	0.10565	3.55	0.00019262	5.80	3.3157×10^{-9}	8.05	4.1397×10^{-16}
1.30	0.0968	3.60	0.00015911	5.85	2.4579×10^{-9}	8.10	2.748×10^{-16}
1.35	0.088508	3.65	0.00013112	5.90	1.8175×10^{-9}	8.15	1.8196×10^{-16}
1.40	0.080757	3.70	0.0001078	5.95	1.3407×10^{-9}	8.20	1.2019×10^{-16}
1.45	0.073529	3.75	8.8417×10^{-5}	6.00	9.8659×10^{-10}	8.25	7.9197×10^{-17}
1.50	0.066807	3.80	7.2348×10^{-5}	6.05	7.2423×10^{-10}	8.30	5.2056×10^{-17}
1.55	0.060571	3.85	5.9059×10^{-5}	6.10	5.3034×10^{-10}	8.35	3.4131×10^{-17}
1.60	0.054799	3.90	4.8096×10^{-5}	6.15	3.8741×10^{-10}	8.40	2.2324×10^{-17}
1.65	0.049471	3.95	3.9076×10^{-5}	6.20	2.8232×10^{-10}	8.45	1.4565×10^{-17}
1.70	0.044565	4.00	3.1671×10^{-5}	6.25	2.0523×10^{-10}	8.50	9.4795×10^{-18}
1.75	0.040059	4.05	2.5609×10^{-5}	6.30	1.4882×10^{-10}	8.55	6.1544×10^{-18}
1.80	0.03593	4.10	2.0658×10^{-5}	6.35	1.0766×10^{-10}	8.60	3.9858×10^{-18}
1.85	0.032157	4.15	1.6624×10^{-5}	6.40	7.7688×10^{-11}	8.65	2.575×10^{-18}
1.90	0.028717	4.20	1.3346×10^{-5}	6.45	5.5925×10^{-11}	8.70	1.6594×10^{-18}
1.95	0.025588	4.25	1.0689×10^{-5}	6.50	4.016×10^{-11}	8.75	1.0668×10^{-18}
2.00	0.02275	4.30	8.5399×10^{-6}	6.55	2.8769×10^{-11}	8.80	6.8408×10^{-19}
2.05	0.020182	4.35	6.8069×10^{-6}	6.60	2.0558×10^{-11}	8.85	4.376×10^{-19}
2.10	0.017864	4.40	5.4125×10^{-6}	6.65	1.4655×10^{-11}	8.90	2.7923×10^{-19}
2.15	0.015778	4.45	4.2935×10^{-6}	6.70	1.0421×10^{-11}	8.95	1.7774×10^{-19}
2.20	0.013903	4.50	3.3977×10^{-6}	6.75	7.3923×10^{-12}	9.00	1.1286×10^{-19}
2.25	0.012224						

Adapted from table 6.1 M F Mesiya - Contemporary Communication Systems

**The value of $\mathrm{erf}(3.30)$ should be ≈ 0.999997 instead, but this value is quoted in the formula table.

Receiver output shit

$$r_o(t) = \begin{cases} s_{o1}(t) + n_o(t) & \text{code 1} \\ s_{o2}(t) + n_o(t) & \text{code 0} \end{cases}$$
$$n : \text{AWGN with } \sigma_o^2$$

ISI, channel model

Nyquist criterion for zero ISI

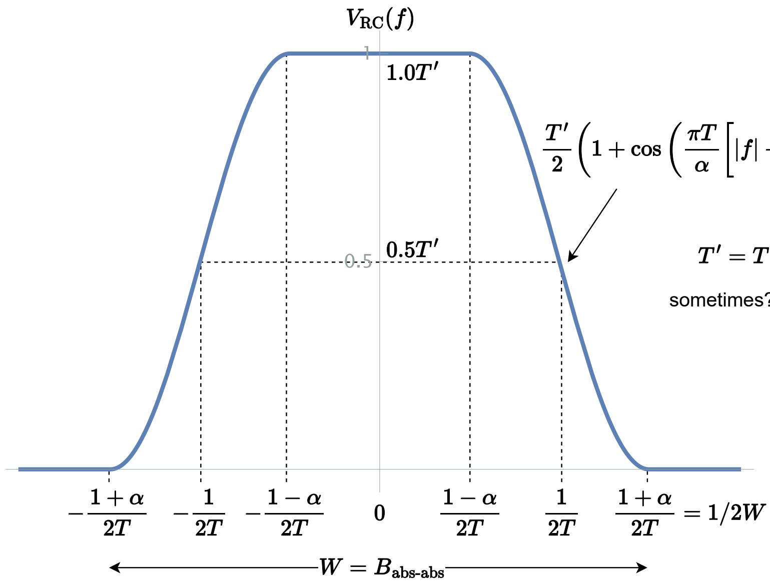
TODO:

Nomenclature

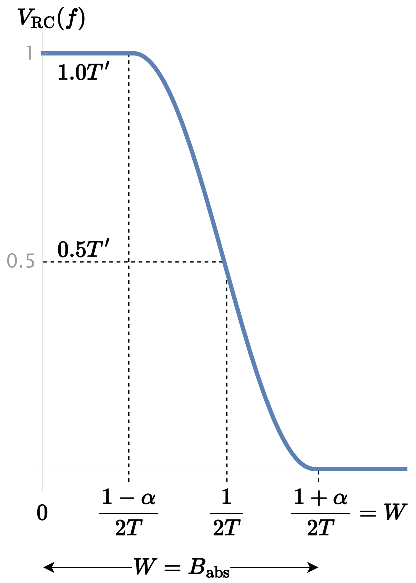
$$\begin{aligned} D &\to \text{Symbol Rate, Max. Signalling Rate} \\ T &\to \text{Symbol Duration} \\ M &\to \text{Symbol set size} \\ W &\to \text{Bandwidth} \end{aligned}$$

Raised cosine (RC) pulse

Linear modulation RC pulse



NRZ unipolar RC pulse



$$0 \leq \alpha \leq 1$$

⚠ NOTE might not be safe to assume $T'=T$, if you can solve the question without T then use that method.

To solve this type of question:

1. Use the formula for D below
2. Consult the BER table below to get the BER which relates the noise of the channel N_0 to E_b and to R_b.

Linear modulation (M-PSK, M-QAM)	**NRZ unipolar encoding**
$W=B_{\text{abs-abs}}$	$W=B_{\text{abs}}$
$W=B_{\text{abs-abs}}=\frac{1+\alpha}{T}=(1+\alpha)D$	$W=B_{\text{abs}}=\frac{1+\alpha}{2T}=(1+\alpha)D/2$
$D=\frac{W\text{ symbol/s}}{1+\alpha}$	$D=\frac{2W\text{ symbol/s}}{1+\alpha}$

Symbol set size M

$$D\text{ symbol/s}=\frac{2W\text{ Hz}}{1+\alpha}$$
$$R_b\text{ bit/s}=(D\text{ symbol/s})\times(k\text{ bit/symbol})$$
$$M\text{ symbol/set}=2^k$$
$$E_b=PT=P_{\text{av}}/R_b\quad\text{Energy per bit}$$

Nyquist stuff

Condition for 0 ISI TODO:

$$P_r(kT)=\begin{cases}1 & k=0\\ 0 & k\neq 0\end{cases}$$

Other

$$\text{Excess BW}=B_{\text{abs}}-B_{\text{Nyquist}}=\frac{1+\alpha}{2T}-\frac{1}{2T}=\frac{\alpha}{2T}\quad\text{FOR NRZ (Use correct }B_{\text{abs}})$$
$$\alpha=\frac{\text{Excess BW}}{B_{\text{Nyquist}}}=\frac{B_{\text{abs}}-B_{\text{Nyquist}}}{B_{\text{Nyquist}}}$$
$$T=1/D$$

Table of bandpass signalling and BER

Binary Bandpass Signaling	$B_{\text{null-null}}$ **(Hz)**	$B_{\text{abs-abs}} = 2B_{\text{abs}}$ **(Hz)**	**BER with Coherent Detection**	**BER with Noncoherent Detection**
ASK, unipolar NRZ	$2R_b$	$R_b(1+\alpha)$	$Q\left(\sqrt{E_b/N_0}\right)$	$0.5\exp(-E_b/(2N_0))$
BPSK	$2R_b$	$R_b(1+\alpha)$	$Q\left(\sqrt{2E_b/N_0}\right)$	Requires coherent detection
Sunde's FSK	$3R_b$		$Q\left(\sqrt{E_b/N_0}\right)$	$0.5\exp(-E_b/(2N_0))$
DBPSK, M-ary Bandpass Signaling	$2R_b$	$R_b(1+\alpha)$		$0.5\exp(-E_b/N_0)$
QPSK/ OQPSK ($M=4$, **PSK**)	R_b	$\frac{R_b(1+\alpha)}{2}$	$Q\left(\sqrt{2E_b/N_0}\right)$	Requires coherent detection
MSK	$1.5R_b$	$\frac{3R_b(1+\alpha)}{4}$	$Q\left(\sqrt{2E_b/N_0}\right)$	Requires coherent detection
M-PSK ($M>4$)	$2R_b/\log_2 M$	$\frac{R_b(1+\alpha)}{\log_2 M}$	$\frac{2}{\log_2 M}Q\left(\sqrt{2\log_2 M\sin^2(\pi/M)\,E_b/N_0}\right)$	Requires coherent detection
M-DPSK ($M>4$)	$2R_b/\log_2 M$	$\frac{R_b(1+\alpha)}{2\log_2 M}$		$\frac{2}{\log_2 M}Q\left(\sqrt{4\log_2 M\sin^2(\pi/(2M))\,E_b/N_0}\right)$
M-QAM (Square constellation)	$2R_b/\log_2 M$	$\frac{R_b(1+\alpha)}{\log_2 M}$	$\frac{4}{\log_2 M}\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{3\log_2 M}{M-1}E_b/N_0}\right)$	Requires coherent detection
M-FSK Coherent	$\frac{(M+3)R_b}{2\log_2 M}$		$\frac{M-1}{\log_2 M}Q\left(\sqrt{(\log_2 M)E_b/N_0}\right)$	
Noncoherent	$2MR_b/\log_2 M$			$\frac{M-1}{2\log_2 M}0.5\exp(-(\log_2 M)E_b/2N_0)$

Adapted from table 11.4 M F Mesiya - Contemporary Communication Systems

PSD of modulated signals

Modulation	$G_x(f)$
Quadrature	$\frac{A_c^2}{4}[G_I(f-f_c)+G_I(f+f_c)+G_Q(f-f_c)+G_Q(f+f_c)]$
Linear	$\frac{\lvert V(f)\rvert^2}{2}\sum_{l=-\infty}^{\infty}R(l)\exp(-j2\pi lfT)\quad\text{What??}$

Symbol error probability

- Minimum distance between any two point
- Different from bit error since a symbol can contain multiple bits

Information theory

Entropy for discrete random variables

$$
\begin{aligned}
H(x) &\geq 0 \\
H(x) &= -\sum_{x_i \in A_x} p_X(x_i) \log_2(p_X(x_i)) \\
H(x,y) &= -\sum_{x_i \in A_x} \sum_{y_i \in A_y} p_{XY}(x_i, y_i) \log_2(p_{XY}(x_i, y_i)) \quad \text{Joint entropy} \\
H(x,y) &= H(x) + H(y) \quad \text{Joint entropy if } x \text{ and } y \text{ independent} \\
H(x|y=y_j) &= -\sum_{x_i \in A_x} p_X(x_i|y=y_j) \log_2(p_X(x_i|y=y_j)) \quad \text{Conditional entropy} \\
H(x|y) &= -\sum_{y_j \in A_y} p_Y(y_j) H(x|y=y_j) \quad \text{Average conditional entropy, equivocation} \\
H(x|y) &= -\sum_{x_i \in A_x} \sum_{y_i \in A_y} p_X(x_i, y_j) \log_2(p_X(x_i|y=y_j)) \\
H(x|y) &= H(x,y) - H(y) \\
H(x,y) &= H(x) + H(y|x) = H(y) + H(x|y)
\end{aligned}
$$

Entropy is **maximized** when all have an equal probability.

Differential entropy for continuous random variables

TODO: Cut out if not required

$$
h(x) = -\int_{\mathbb{R}} f_X(x) \log_2(f_X(x)) dx
$$

Mutual information

Amount of entropy decrease of x after observation by y.

$$
I(x;y) = H(x) - H(x|y) = H(y) - H(y|x)
$$

Channel model

Vertical, x: input
Horizontal, y: output

$$
\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1N} \\ p_{21} & p_{22} & \cdots & p_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{M1} & p_{M2} & \cdots & p_{MN} \end{bmatrix}
$$

$P(y_j\|x_i)$	y_1	y_2	$\cdots$	y_N
x_1	p_{11}	p_{12}	$\cdots$	p_{1N}
x_2	p_{21}	p_{22}	$\cdots$	p_{2N}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_M	p_{M1}	p_{M2}	$\cdots$	p_{MN}

Input has probability distribution $p_X(a_i) = P(X = a_i)$

Channel maps alphabet ‘$\{a_1, \ldots, a_M\} \to \{b_1, \ldots, b_N\}$‘

Output has probabiltiy distribution $p_Y(b_j) = P(y = b_j)$

$$
\begin{aligned}
p_Y(b_j) &= \sum_{i=1}^{M} P[x = a_i, y = b_j] \quad 1 \leq j \leq N \\
&= \sum_{i=1}^{M} P[X = a_i] P[Y = b_j | X = a_i] \\
[p_Y(b_0) \quad p_Y(b_1) \quad \ldots \quad p_Y(b_j)] &= [p_X(a_0) \quad p_X(a_1) \quad \ldots \quad p_X(a_i)] \times \mathbf{P}
\end{aligned}
$$

Fast procedure to calculate $I(y;x)$

1. Find $H(x)$
2. Find $[p_Y(b_0) \quad p_Y(b_1) \quad \ldots \quad p_Y(b_j)] = [p_X(a_0) \quad p_X(a_1) \quad \ldots \quad p_X(a_i)] \times \mathbf{P}$
3. Multiply each row in $\mathbf{P}$ by $p_X(a_i)$ since $p_{XY}(x_i, y_i) = P(y_i|x_i)P(x_i)$
4. Find $H(x,y)$ using each element from (3.)
5. Find $H(x|y) = H(x,y) - H(y)$
6. Find $I(y;x) = H(x) - H(x|y)$

Channel types

Type	Definition
Symmetric channel	Every row is a permutation of every other row, Every column is a permutation of every other column. $\text{Symmetric} \implies \text{Weakly symmetric}$
Weakly symmetric	Every row is a permutation of every other row, Every column has the same sum

Channel capacity of weakly symmetric channel

$$\begin{aligned} C &\to \text{Channel capacity (bits/channels used)} \\ N &\to \text{Output alphabet size} \\ \mathbf{p} &\to \text{Probability vector, any row of the transition matrix} \\ C &= \log_2(N) - H(\mathbf{p}) \quad \text{Capacity for weakly symmetric and symmetric channels} \\ R &< C \text{ for error-free transmission} \end{aligned}$$

Channel capacity of an AWGN channel

$$y_i = x_i + n_i \quad n_i \sim N(0, N_0/2)$$

$$C = \frac{1}{2} \log_2 \left(1 + \frac{P_{\text{av}}}{N_0/2}\right)$$

Channel capacity of a bandwidth AWGN channel

Note: Define XOR ($\oplus$) as exclusive OR, or modulo-2 addition.

$$\begin{aligned} &P_s \to \text{Bandwidth limited average power} \\ &y_i = \text{bandpass}_W(x_i) + n_i \quad n_i \sim N(0, N_0/2) \\ &C = W \log_2 \left(1 + \frac{P_s}{N_0 W}\right) \\ &C = W \log_2(1 + \text{SNR}) \quad \text{SNR} = P_s/(N_0 W) \end{aligned}$$

Channel code

Hamming weight	$w_H(x)$	Number of `'1'` in codeword x
Hamming distance	$d_H(x_1, x_2) = w_H(x_1 \oplus x_2)$	Number of different bits between codewords x_1 and x_2 which is the hamming weight of the XOR of the two codes.
Minimum distance	$d_{\min}$	**IMPORTANT**: $x \neq \mathbf{0}$, excludes weight of all-zero codeword. For a linear block code, $d_{\min} = w_{\min}$

Linear block code

Code is (n, k)

n is the width of a codeword

2^k codewords

A linear block code must be a subspace and satisfy both:

1. Zero vector must be present at least once
2. The XOR of any codeword pair in the code must result in a codeword that is already present in the code table.

For a linear block code, $d_{\min} = w_{\min}$

Code generation

Each generator vector is a binary string of size n. There are k generator vectors in $\mathbf{G}$.

$$
\begin{aligned}
\mathbf{g}_i &= [g_{i,0} \quad \cdots \quad g_{i,n-2} \quad g_{i,n-1}] \\
\mathbf{g}_0 &= [1010] \quad \text{Example for } n = 4 \\
\mathbf{G} &= \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_{k-1} \end{bmatrix} = \begin{bmatrix} g_{0,0} & \cdots & g_{0,n-2} & g_{0,n-1} \\ g_{1,0} & \cdots & g_{1,n-2} & g_{1,n-1} \\ \vdots & \ddots & \vdots & \vdots \\ g_{k-1,0} & \cdots & g_{k-1,n-2} & g_{k-1,n-1} \end{bmatrix}
\end{aligned}
$$

A message block $\mathbf{m}$ is coded as $\mathbf{x}$ using the generation codewords in $\mathbf{G}$:

$$
\begin{aligned}
\mathbf{m} &= [m_0 \quad \dots \quad m_{n-2} \quad m_{k-1}] \\
\mathbf{m} &= [101001] \quad \text{Example for } k = 6 \\
\mathbf{x} &= \mathbf{mG} = m_0\mathbf{g}_0 + m_1\mathbf{g}_1 + \cdots + m_{k-1}\mathbf{g}_{k-1}
\end{aligned}
$$

Systemic linear block code

Contains k message bits (Copy $\mathbf{m}$ as-is) and $(n-k)$ parity bits after the message bits.

$$
\begin{aligned}
\mathbf{G} = \quad & [\mathbf{I}_k \mid \mathbf{P}] = \left[\begin{array}{cccc|cccc} 1 & 0 & \dots & 0 & p_{0,0} & \cdots & p_{0,n-2} & p_{0,n-1} \\ 0 & 1 & \dots & 0 & p_{1,0} & \cdots & p_{1,n-2} & p_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & p_{k-1,0} & \cdots & p_{k-1,n-2} & p_{k-1,n-1} \end{array}\right] \\
\mathbf{m} &= [m_0 \quad \dots \quad m_{n-2} \quad m_{k-1}] \\
\mathbf{x} &= \mathbf{mG} = \mathbf{m}\ [\mathbf{I}_k \mid \mathbf{P}] = [\mathbf{mI}_k \mid \mathbf{mP}] = [\mathbf{m} \mid \mathbf{b}] \\
\mathbf{b} &= \mathbf{mP} \quad \text{Parity bits of } \mathbf{x}
\end{aligned}
$$

Parity check matrix $\mathbf{H}$

Transpose $\mathbf{P}$ for the parity check matrix

$$
\begin{aligned}
\mathbf{H} &= [\mathbf{P}^{\mathrm{T}} \mid \mathbf{I}_{n-k}] \\
&= [\ \mathbf{p}_0{}^{\mathrm{T}} \quad \mathbf{p}_1{}^{\mathrm{T}} \quad \dots \quad \mathbf{p}_{k-1}{}^{\mathrm{T}} \mid \mathbf{I}_{n-k}\ ] \\
&= \left[\begin{array}{cccc|cccc} p_{0,0} & \cdots & p_{0,k-2} & p_{0,k-1} & 1 & 0 & \dots & 0 \\ p_{1,0} & \cdots & p_{1,k-2} & p_{1,k-1} & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1,0} & \cdots & p_{n-1,k-2} & p_{n-1,k-1} & 0 & 0 & \dots & 1 \end{array}\right] \\
\mathbf{xH}^{\mathrm{T}} &= \mathbf{0} \implies \text{Codeword is valid}
\end{aligned}
$$

Procedure to find parity check matrix from list of codewords

1. From the number of codewords, find $k = \log_2(N)$
2. Partition codewords into k information bits and remaining bits into $n-k$ parity bits. The information bits should be a simple counter (?).
3. Express parity bits as a linear combination of information bits
4. Put coefficients into $\mathbf{P}$ matrix and find $\mathbf{H}$

Example:

x_1	x_2	x_3	x_4	x_5
1	0	1	1	0
0	1	1	1	1
0	0	0	0	0
1	1	0	0	1

Set x_1, x_2 as information bits. Express x_3, x_4, x_5 in terms of x_1, x_2.

$$
\begin{aligned}
x_3 &= x_1 \oplus x_2 \\
x_4 &= x_1 \oplus x_2 \implies \mathbf{P} = \begin{array}{c|cc} & x_1 & x_2 \\ \hline x_3 & 1 & 1 \\ x_4 & 1 & 1 \\ x_5 & 0 & 1 \end{array} \\
x_5 &= x_2
\end{aligned}
$$

$$
\mathbf{H} = \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right]
$$

Error detection and correction

Detection of s errors: $d_{\text{min}} \geq s + 1$

Correction of u errors: $d_{\text{min}} \geq 2u + 1$

CHECKLIST

- Transfer function in complex envelope form $\tilde{h}(t)$ should be divided by two.
- Convolutions: do not forget width when using graphical method
- todo: add more items to check